Πρόταση: Αν $X \cup Y = Y$ τότε $X \subseteq Y$.

Απόδειξη: $(x \in X) \Rightarrow (x \in Y) \equiv \neg(x \in X) \vee (x \in Y)$

$$\equiv \neg(x \in X) \vee (x \in X \cup Y)$$

$$\equiv \neg(x \in X) \vee ((x \in X) \vee (x \in Y))$$

$$\equiv (\neg(x \in X) \vee (x \in X)) \vee (x \in Y)$$

$$\equiv \mathbb{1} \vee (x \in Y) \equiv \mathbb{1}$$

Άρα $(x \in X) \Rightarrow (x \in Y) \equiv \mathbb{1}$

Άρα $X \subseteq Y$.

Made with Goodnotes

Άσκηση 2B: $A \cup B = A \cap B$ αν και μόνο αν $A = B$

Απόδειξη: Αν $A = B$ τότε
$$\left.\begin{aligned} A \cap B &= A \cap A = A \\ A \cup B &= A \cup A = A \end{aligned}\right\} \leadsto A \cup B = A \cap B$$

Έστω $A \cup B = A \cap B$

$$\left.\begin{aligned} &\left.\begin{aligned} &A \cup B = A \cap B \\ \text{CW 5} \rightarrow\ &A \cap B \subseteq A \end{aligned}\right\} \leadsto A \cup B \subseteq A \\ &\text{CW 5} \quad A \subseteq A \cup B \end{aligned}\right\} \leadsto A \cup B = A \overset{\text{Πρόταση}}{\leadsto} B \subseteq A$$

$$\left.\begin{aligned} &\left.\begin{aligned} &A \cup B = A \cap B \\ \text{CW 5} \rightarrow\ &A \cap B \subseteq B \end{aligned}\right\} \leadsto A \cup B \subseteq B \\ &\text{CW 5} \quad B \subseteq A \cup B \end{aligned}\right\} \leadsto A \cup B = B \overset{\text{Πρόταση}}{\leadsto} A \subseteq B$$

$$\left.\begin{aligned} B \subseteq A \\ A \subseteq B \end{aligned}\right\} A = B$$

Alternative solution 2B

Αν $A \cup B = A \cap B$ τότε $A = B$

$$x \in A \equiv (x \in A) \wedge 1 \equiv (x \in A) \wedge \big((x \in B) \vee \neg(x \in B)\big)$$

$$\equiv \big((x \in A) \wedge (x \in B)\big) \vee \big((x \in A) \wedge \neg(x \in B)\big)$$

$$\equiv (x \in A \cap B) \vee (x \in A \setminus B)$$

$(*)$ $A \cap B = A \cup B$

$$\overset{(*)}{\equiv} (x \in A \cup B) \vee (x \in A \setminus B)$$

$$\equiv x \in (A \cup B) \cup (A \setminus B)$$

$(*)$ $A \setminus B \subseteq A \subseteq A \cup B$

Άρα $(A \cup B) \cup (A \setminus B) = A \cup B$

$$\overset{(*)}{\equiv} x \in A \cup B$$

Άρα $\boxed{x \in A \equiv x \in A \cup B}$ (1). Όμοια $\boxed{x \in B \equiv x \in A \cup B}$ (2)

1, 2 $\leadsto$ $x \in A \equiv x \in B$ Άρα $A = B$.

Άσκηση 25: $A \setminus B = B \setminus A$ αν και μόνο αν $A = B$.

<u>Απόδειξη</u>: Αν $A = B$ τότε $A \setminus B = A \setminus A = B \setminus A$.

<u>Απόδειξη</u>: Αν $\boxed{\circledast \; A \setminus B = B \setminus A}$

τότε $A \overset{\circledast}{=} (\underline{A \setminus B}) \cup (A \cap B) \overset{\circledast}{=} (\underline{B \setminus A}) \cup (A \cap B) \overset{\circledast}{=} B$.

Άρα $A = B$

$\circledast \quad \underline{X = (X \setminus Y) \cup (X \cap Y)}$

$$
\begin{aligned}
x \in X &\equiv (x \in X) \wedge \mathbb{1} \\
&\equiv (x \in X) \wedge (\neg(x \in Y) \vee (x \in Y)) \\
&\equiv (x \in X \wedge \neg(x \in Y)) \vee ((x \in X) \wedge (x \in Y)) \\
&\equiv (x \in X \setminus Y) \vee (x \in X \cap Y) \\
&\equiv x \in (X \setminus Y) \cup (X \cap Y)
\end{aligned}
$$

Άσκηση 2Δ: $A \triangle B = B \setminus A$ αν και μόνο αν $A \subseteq B$.

Απόδειξη: Αν $A \subseteq B$ τότε $x \in A \setminus B = (x \in A) \wedge \neg(x \in B) \equiv 0$

	$x \in A$	$x \in B$	$\neg(x \in B)$	$(x\in A) \wedge \neg(x \in B)$
	1	1	0	0
	0	0	1	0
$A \subseteq B \rightarrow$	~~1~~	~~0~~	~~1~~	~~1~~
	0	1	0	0

Άρα $A \setminus B = \emptyset$

Άρα $A \triangle B = (A \setminus B) \cup (B \setminus A) = \emptyset \cup (B \setminus A) = B \setminus A$.

Άρα $A \triangle B = B \setminus A$

Απόδειξη (αντίστροφο)

$X = A \setminus B$
$Y = B \setminus A$

Αν $A \triangle B = B \setminus A$ τότε $(A \setminus B) \cup (B \setminus A) = B \setminus A \overset{\text{Πρόταση}}{\rightsquigarrow} A \setminus B \subseteq B \setminus A$

$A \setminus B \subseteq B \setminus A \rightsquigarrow (A \setminus B) \cup B \subseteq (B \setminus A) \cup B \rightsquigarrow A \cup B \subseteq B$

CW 15 $B \subseteq A \cup B$

$\rightarrow A \cup B = B$

Πρόταση

$A \subseteq B$