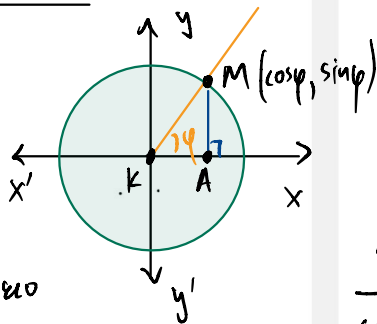


Τριγωνομετρικές Ταυτότητες

$$\cos^2 \varphi + \sin^2 \varphi = 1$$

Απόδειξη

Φέρω $MA \perp x'$



Από το Πυθαγόρειο
Θεώρημα έχω

($\triangle KAM$)

$$|MA|^2 + |KA|^2 = |KM|^2 \Leftrightarrow$$

$$(\sin \varphi)^2 + (\cos \varphi)^2 = 1^2 \Leftrightarrow$$

$$\sin^2 \varphi + \cos^2 \varphi = 1$$

$$\tan^2 \varphi + 1 = \sec^2 \varphi$$

$$\varphi \neq \frac{\pi}{2}, \frac{3\pi}{2}$$

Απόδειξη:

$$\sin^2 \varphi + \cos^2 \varphi = 1 \Leftrightarrow$$

$$\frac{1}{\cos^2 \varphi} \cdot (\sin^2 \varphi + \cos^2 \varphi) = \frac{1}{\cos^2 \varphi} \cdot 1 \Leftrightarrow$$

$$\frac{\sin^2 \varphi}{\cos^2 \varphi} + \frac{\cos^2 \varphi}{\cos^2 \varphi} = \frac{1}{\cos^2 \varphi} \Leftrightarrow$$

$$\left(\frac{\sin \varphi}{\cos \varphi}\right)^2 + 1 = \left(\frac{1}{\cos \varphi}\right)^2 \Leftrightarrow$$

$$\tan^2 \varphi + 1 = \sec^2 \varphi$$

$$\cot^2 \varphi + 1 = \csc^2 \varphi$$

$$\varphi \neq 0, \pi$$

Απόδειξη: $\cos^2 \varphi + \sin^2 \varphi = 1 \Leftrightarrow$

$$\frac{1}{\sin^2 \varphi} (\cos^2 \varphi + \sin^2 \varphi) = \frac{1}{\sin^2 \varphi} \cdot 1 \Leftrightarrow$$

$$\frac{\cos^2 \varphi}{\sin^2 \varphi} + \frac{\cancel{\sin^2 \varphi}}{\cancel{\sin^2 \varphi}} = \frac{1}{\sin^2 \varphi} \Leftrightarrow$$

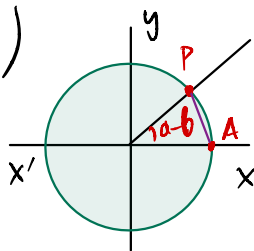
$$\left(\frac{\cos \varphi}{\sin \varphi} \right)^2 + 1 = \left(\frac{1}{\sin \varphi} \right)^2$$

$$\cot^2 \varphi + 1 = \csc^2 \varphi$$

$$\cos(a-b) = \cos a \cdot \cos b + \sin a \cdot \sin b$$

$$P(\cos(a-b), \sin(a-b))$$

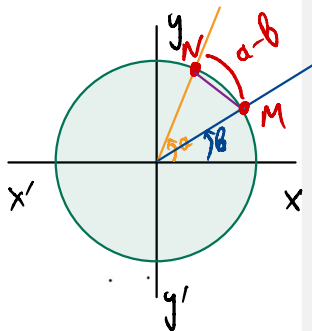
$$A(1, 0)$$



Απόδειξη:

$$M(\cos b, \sin b)$$

$$N(\cos a, \sin a)$$



$$|PA| = \sqrt{(\cos(a-b) - 1)^2 + (\sin(a-b) - 0)^2}$$

$$= \sqrt{\cos^2(a-b) + 1 - 2\cos(a-b) + \sin^2(a-b)}$$

$$= \sqrt{\underbrace{\cos^2(a-b) + \sin^2(a-b)}_1 + 1 - 2\cos(a-b)}$$

$$= \sqrt{2 - 2\cos(a-b)}$$

$$|MN| = \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2}$$

$$= \sqrt{\cos^2 a + \cos^2 b - 2\cos a \cdot \cos b + \sin^2 a + \sin^2 b - 2\sin a \cdot \sin b}$$

$$= \sqrt{(\cos^2 a + \sin^2 a) + (\cos^2 b + \sin^2 b) - 2\cos a \cdot \cos b - 2\sin a \cdot \sin b}$$

$$= \sqrt{1 + 1 - 2\cos a \cdot \cos b - 2\sin a \cdot \sin b}$$

$$|MN| \stackrel{?}{=} |PA|$$

$$(a-b)$$

$$|MN| = |PA|$$

$$\Rightarrow \sqrt{2 - 2 \cos a \cdot \cos b - 2 \sin a \cdot \sin b} = \sqrt{2 - 2 \cos(a-b)}$$

$$\Rightarrow 2 - 2 \cos a \cdot \cos b - 2 \sin a \cdot \sin b = 2 - 2 \cos(a-b)$$

$$\Rightarrow -2 \cos a \cdot \cos b - 2 \sin a \cdot \sin b = -2 \cos(a-b)$$

$$\Rightarrow \cos a \cdot \cos b + \sin a \cdot \sin b = \cos(a-b)$$

$$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$

Απόδειξη:

$$\begin{aligned}\cos(a+b) &= \cos(a - (-b)) \\ &= \cos a \cdot \cos(-b) + \sin a \cdot \sin(-b) \\ &= \cos a \cdot \cos b + \sin a \cdot (-\sin b) \\ &= \cos a \cdot \cos b - \sin a \cdot \sin b\end{aligned}$$

Ή αλλιώς,

$$\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$$

$$\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$$

Απόδειξη:

$$\begin{aligned}\sin(a-b) &= \cos\left(\frac{\pi}{2} - (a-b)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - a\right) + b\right) \\ &= \cos\left(\frac{\pi}{2} - a\right) \cdot \cos b - \sin\left(\frac{\pi}{2} - a\right) \cdot \sin b \\ &= \sin a \cdot \cos b - \cos a \cdot \sin b\end{aligned}$$

Ή αλλιώς,

$$\sin(a-b) = \sin a \cdot \cos b - \cos a \cdot \sin b$$

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

Απόδειξη:

$$\begin{aligned}\sin(a+b) &= \sin(a - [-b]) \\ &= \sin a \cdot \cos(-b) - \cos a \cdot \sin(-b) \\ &= \sin a \cdot \cos b - \cos a \cdot (-\sin b) \\ &= \sin a \cdot \cos b + \cos a \cdot \sin b\end{aligned}$$

Άρα,

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b$$

$$\sin(2a) = 2 \cdot \sin a \cdot \cos a$$

Απόδειξη:

$$\begin{aligned}\sin 2a &= \sin(a+a) \\ &= \sin a \cdot \cos a + \cos a \cdot \sin a \\ &= \sin a \cdot \cos a + \sin a \cdot \cos a \\ &= 2 \cdot \sin a \cdot \cos a\end{aligned}$$

