12. 1. X • (Y ∨ Z) ∴ (X • Y) ∨ Z
2. (X • Y) ∨ (X • Z)
3. [(X • Y) ∨ X] • [(X • Y) ∨ Z]
4. (X • Y) ∨ Z

* 13. 1. ~P → P ∴ ~P → Q
2. ~~P ∨ P
3. ~~P ∨ ~~P
4. ~~P
5. ~~P ∨ Q
6. ~P → Q

14. 1. [(~R ∨ ~S) • ~S] ∨ [R • (~R ∨ ~S)] ∴ ~R → ~S
2. [(~R ∨ ~S) • ~S] ∨ [(~R ∨ ~S) • R]
3. (~R ∨ ~S) • (~S ∨ R)
4. ~S ∨ R
5. S → R
6. ~R → ~S

15. 1. Q ∨ (~P • T) ∴ (~Q → ~P) • (~Q → T)
2. (Q ∨ ~P) • (Q ∨ T)
3. Q ∨ ~P
4. ~P ∨ Q
5. P → Q
6. ~Q → ~P
7. Q ∨ T
8. ~~Q ∨ T
9. ~Q → T
10. (~Q → ~P) • (~Q → T)

Part B: Correct or Incorrect? Some of the following inferences are correct applications of our rules, and some are not. If an inference is a correct application of our rules, name the rule. If an inference is not a correct application of our rules, explain why it is not. (The question is whether the conclusion in each case can be reached *in a single step* from the premise by an application of one of our rules.)

* 1. (~B ∨ ~B) ↔ A
∴ ~B ↔ A

2. N ∨ M
∴ ~N → M

3. ~~P ∨ Q
∴ ~P → Q

* 4. (C • ~L) ∨ (C • S)
∴ C • (~L ∨ S)

5. (~M • N) → ~L
∴ ~M → (N → ~L)

6. S → (R • R)
∴ S → R

* 7. K ∨ (X • R)
∴ (K • X) ∨ (K • R)

8. ~H → P
∴ H ∨ P

9. $X \rightarrow (Y \rightarrow Z)$
$\therefore (X \bullet Y) \rightarrow Z$

* 10. $({\sim}U \vee S) \rightarrow Q$
${\sim}{\sim}U$
$\therefore S \rightarrow Q$

11. $({\sim}W \vee {\sim}U) \rightarrow ({\sim}F \vee {\sim}F)$
$\therefore ({\sim}W \vee {\sim}U) \rightarrow {\sim}F$

12. ${\sim}A \rightarrow {\sim}B$
$\therefore {\sim}{\sim}A \vee {\sim}B$

* 13. ${\sim}A \vee (N \bullet Z)$
$\therefore ({\sim}A \vee N) \bullet ({\sim}A \vee Z)$

14. $F \rightarrow (G \bullet H)$
$\therefore F \rightarrow (G \rightarrow H)$

15. $({\sim}J \bullet {\sim}K) \vee ({\sim}J \bullet {\sim}L)$
$\therefore {\sim}J \bullet ({\sim}K \vee {\sim}L)$

* 16. $M \bullet (O \vee U)$
$\therefore (M \bullet O) \vee (M \bullet U)$

17. $(S \vee T) \bullet (S \vee {\sim}W)$
$\therefore S \vee (T \bullet {\sim}W)$

18. $A \bullet ({\sim}B \vee C)$
$\therefore (A \vee {\sim}B) \bullet (A \vee C)$

* 19. $(E \bullet H) \rightarrow V$
$\therefore E$

20. $(B \rightarrow C) \vee K$
B
$\therefore C \vee K$

Part C: Short Proofs Construct proofs for each of the following symbolic arguments.

* 1. ${\sim}M \vee N \;\; \therefore {\sim}N \rightarrow {\sim}M$

2. ${\sim}B \leftrightarrow C,\; {\sim}B \;\; \therefore C$

3. ${\sim}S \vee (R \bullet T),\; {\sim}R \;\; \therefore {\sim}S$

* 4. ${\sim}A \vee {\sim}A,\; A \vee P \;\; \therefore P$

5. $(D \rightarrow C) \bullet (C \rightarrow D),\; E \rightarrow {\sim}(D \leftrightarrow C) \;\; \therefore {\sim}E$

6. $F \bullet (G \vee H),\; {\sim}F \vee {\sim}H \;\; \therefore F \bullet G$

* 7. $({\sim}J \bullet K) \rightarrow L,\; {\sim}J \;\; \therefore {\sim}L \rightarrow {\sim}K$

8. $({\sim}N \vee M) \bullet ({\sim}N \vee O) \;\; \therefore {\sim}(M \bullet O) \rightarrow {\sim}N$

9. $P \bullet P,\; Q \rightarrow {\sim}P \;\; \therefore {\sim}Q$

* 10. ${\sim}R,\; (R \rightarrow S) \rightarrow T \;\; \therefore T$

11. $U \rightarrow (X \rightarrow W),\; Z \rightarrow {\sim}[(U \bullet X) \rightarrow W] \;\; \therefore {\sim}Z$

12. ${\sim}D \;\; \therefore C \rightarrow {\sim}D$

* 13. $E \rightarrow H,\; [(E \vee F) \bullet (E \vee G)],\; [(F \bullet G) \rightarrow H] \;\; \therefore H$

14. $({\sim}J \bullet K) \vee ({\sim}J \bullet L),\; M \rightarrow J \;\; \therefore {\sim}M$

15. ${\sim}N \leftrightarrow {\sim}O,\; ({\sim}O \rightarrow {\sim}N) \rightarrow P \;\; \therefore P$

* 16. ${\sim}{\sim}(R \bullet S),\; T \rightarrow (R \rightarrow {\sim}S) \;\; \therefore {\sim}T$

17. ${\sim}C,\; ({\sim}A \bullet B) \vee ({\sim}A \bullet C) \;\; \therefore B$

18. $({\sim}D \vee E) \bullet ({\sim}D \vee {\sim}F),\; (E \bullet {\sim}F) \rightarrow {\sim}G,\; {\sim}D \rightarrow {\sim}G \;\; \therefore {\sim}G$

* 19. $H \vee H,\; H \leftrightarrow {\sim}J \;\; \therefore {\sim}J$

20. X ↔ Y, ~~(X ∨ Y) ∴ X • Y
21. [(L • M) ∨ ~L] • [(L • M) ∨ ~M] ∴ L ↔ M
*22. P • Q ∴ [(R ∨ P) • R] ∨ [(R ∨ P) • Q]
23. R • ~R ∴ S • ~S
24. ~O ∴ ~Q → ~(O • P)
*25. (A → B) ↔ C, ~(A → B) ∨ ~C ∴ ~C

Part D: Longer Proofs Construct a proof for each of the following symbolic arguments.

* 1. (Z ∨ ~Y) • (Z ∨ W), Z → ~~U, ~Y → (W → U) ∴ U
2. ~U → ~B, S → ~B, ~(U • ~S), T ∨ B ∴ T
3. (Q • R) ∨ (~Q • ~R), N → ~(Q ↔ R), E ∨ N ∴ E
* 4. ~H ∨ (G ∨ F), ~F, S → ~(H → G) ∴ ~S
5. ~(J • L), (J → ~L) → (~M • ~X), E ∨ (M ∨ X) ∴ E
6. (L ∨ M) • (L ∨ ~S), A → ~L, A → (~M ∨ S) ∴ ~A
* 7. B ∨ (C • ~D), (D → B) ↔ P ∴ P
8. (G • S) ∨ (G • ~T), ~R → ~G, (T → S) → Q ∴ R • Q
9. ~X ↔ ~Y, ~X ∨ ~Y, Z ↔ Y ∴ ~Z
*10. (B • C) → D, B, Q → ~(~C ∨ D), ~Q ↔ T ∴ T
11. (F • G) ∨ (F • ~H), (H → G) → L, L → (P → ~F) ∴ ~P
12. ~X ∨ (M • O), (X → O) → ~M ∴ ~X
13. (~Z • W) → Q, ~Z, R ↔ (W • ~Q) ∴ ~R
14. A ↔ B, ~~(A ∨ B) ∴ B
15. (A ∨ B) • (A ∨ G), M → ~A, ~Q → (~B ∨ ~G) ∴ M → Q
16. Y • (~N ∨ A), ~Y ∨ N, (A • Y) → ~~K ∴ K
17. ~G → ~F, (~F ∨ G) → (H ∨ J), H → Z, J → ~P ∴ P → Z
18. (D • E) ∨ (~D • ~E), (H • J) → ~(D ↔ E), ~~H ∨ J ∴ J ↔ ~H
19. (~E ∨ Z) • (~E ∨ W), ~K → E, ~K → (~Z ∨ ~W), K ↔ U ∴ R → U
20. (R • S) ∨ (R • ~E), (Y • O) → (E • ~S), (O → ~Y) → L ∴ L
21. ~E, ~(E • D) → F, (~F ∨ B) • (~F ∨ C) ∴ A ∨ (B • C)
22. T → R, R → S, ~R ↔ S ∴ ~T • S
23. (~K → K) → ~L, ~(~L → ~M) → L, M ∴ K ↔ ~L
24. W, ~Y → (~W • ~X) ∴ Y • [(~W • ~X) → Z]
25. F ∨ ~I, I ∨ H, ~(G ↔ J) → ~H ∴ [(~G ∨ ~J) • (G ∨ J)] → F

Part E: English Arguments Symbolize the following arguments, and then construct proofs to show that they are valid.

* 1. If workers should be paid, then either they should be paid according to their needs (as Marx asserted), or they should be paid for services rendered. If workers should be paid according to their needs, then single mothers should be paid more (other things being equal) than their co-workers, and so should workers who have large families. If workers should be paid for services rendered, then workers should receive equal pay for equal work. Workers should be paid, but it is not the case that workers having large families should be paid more (other things being equal) than their co-workers. Hence, workers should receive equal pay for equal work. (P: Workers should be paid; N: Workers should be paid according to their needs; S: Workers should be paid for services rendered; M: Single mothers should be paid more (other things being equal) than their co-workers; F: Workers who have large families should be paid more (other things being equal) than their co-workers; E: Workers should receive equal pay for equal work)

2. If either the defendant refuses to take the stand or he confesses, then he is guilty. We may infer that the defendant is guilty if he refuses to take the stand. (R: The defendant refuses to take the stand; C: The defendant confesses; G: The defendant is guilty)

3. If beauty is in the eye of the beholder, then beauty is not objective. But beauty is objective if it is observable. And beauty can be seen, can't it? Furthermore, beauty can be seen if and only if beauty is observable. Therefore, popular opinion to the contrary, beauty is not in the eye of the beholder. (E: Beauty is in the eye of the beholder; B: Beauty is objective; O: Beauty is observable; S: Beauty can be seen)

* 4. Either sex is for procreation, or it is for interpersonal union and pleasure. If sex is for either procreation or interpersonal union, then societal rules are needed to regulate sex. It follows that societal rules are needed to regulate sex. (S: Sex is for procreation; U: Sex is for interpersonal union; P: Sex is for pleasure; R: Societal rules are needed to regulate sex)

5. Young smokers either identify with their future selves or fail to identify with their future selves. If young smokers identify with their future selves, then they are irrational if they know smoking causes cancer. If young smokers fail to identify with their future selves, then they act without due regard for another person (namely, their future self), assuming that they know smoking causes cancer. And given that young smokers act without due regard for another person, they are immoral. But while young smokers do know that smoking causes cancer, they are not immoral. Therefore, young smokers are irrational, and they identify with their future selves. (I: Young smokers identify with their future selves; R: Young smokers are irrational; K: Young smok-

ers know that smoking causes cancer; A: Young smokers act without due regard for another person; M: Young smokers are immoral) —This argument makes use of material in Derek Parfit, *Reasons and Persons* (New York: Oxford University Press, 1986), pp. 319–320

6. It is a biological fact that animals in most species will make greater sacrifices for near relatives than for others. (For instance, a calf's mother will defend it to the death but will not defend the calf of another cow.) Given this fact, there is a general law that animals act so as to preserve genes similar to their own. But if there is a general law that animals act so as to preserve genes similar to their own, then sociobiologists are right and it is biologically impossible to treat all people equally. Now, if it is biologically impossible to treat all people equally, then it is futile to preach the ideal of equality and futile to preach the ideal of universal love. Hence, it is futile to preach universal love if sociobiologists are right. [*Hint:* In symbolizing the argument, ignore the parenthetical remark.] (B: It is a biological fact that animals in most species will make greater sacrifices for near relatives than for others; G: There is a general law that animals act so as to preserve genes similar to their own; S: Sociobiologists are right; E: It is biologically impossible to treat all people equally; P: It is futile to preach the ideal of equality; U: It is futile to preach the ideal of universal love)

7. You can walk to the door only if you can walk to the halfway point between yourself and the door. But unfortunately, you can walk to the halfway point between yourself and the door only if you can walk to a point *halfway* to the halfway point! Now, if you cannot walk halfway to the halfway point only if you cannot walk to the door, then you can walk to the door only if you can perform an infinite number of acts in a finite period of time. Obviously, you cannot perform an infinite number of acts in a finite period of time. So, as Zeno of Elea concluded, in spite of what your senses may tell you, you cannot walk to the door. (D: You can walk to the door; H: You can walk to the halfway point between yourself and the door; P: You can walk halfway to the halfway point; F: You can perform an infinite number of acts in a finite period of time)

8. This is the best of all possible worlds. For God exists; and if God is not both morally perfect and omnipotent, then God does not exist. Now, if God is omnipotent, God can create just any possible world. And if God is morally perfect, God will create the best possible world if He can create it. And God can create the best of all possible worlds if and only if God can create just any possible world. Moreover, this is the best of all possible worlds given that God will create the best of all possible worlds. (G: God exists; M: God is morally perfect; O: God is omnipotent; A: God can create just any possible world; W: God will create the best possible world; C: God can create the best possible world; B: This is the best of all possible worlds)

9. God cannot know the future free acts of his creatures if God is in time. For if God is in time, God's knowledge of the future is a prediction based on the past and present. However, if humans have free will, then their future acts are not infallibly predictable based on the past and the present. If the future acts of humans are not infallibly predictable based on the past and the present, then God cannot know the future free acts of his creatures if God is in time. Finally, if humans do not have free will, then God's knowledge of the future is not a prediction based on the past and the present. (T: God is in time; P: God's knowledge of the future is a prediction based on the past and present; F: Humans have free will; I: The future acts of humans are infallibly predictable based on the past and present; K: God can know the future free acts of his creatures)

10. All inductive arguments presuppose that the unobserved resembles the observed. (For example, "All observed emeralds have been green; therefore, the next emerald to be found will be green.") Given that all inductive arguments presuppose that the unobserved resembles the observed, induction is unjustified unless we have good reason to believe that the unobserved resembles the observed. If we have good reason to believe that the unobserved resembles the observed, then we have either a good deductive argument or a good inductive argument. We have a good inductive argument only if not all inductive arguments presuppose that the unobserved resembles the observed. We have a good deductive argument only if valid reasoning can begin with the observed and end with the unobserved. Sad to say, valid reasoning cannot begin with the observed and end with the unobserved. It thus appears that David Hume's skeptical conclusion is inescapable: Induction is unjustified. [*Hint:* In symbolizing the argument, ignore the parenthetical remark.] (P: All inductive arguments presuppose that the unobserved resembles the observed; J: Induction is justified; R: We have good reason to believe that the unobserved resembles the observed; D: We have a good deductive argument; I: We have a good inductive argument; V: Valid reasoning can begin with the observed and end with the unobserved)

8.4 Conditional Proof

Consider the following argument.

32. If Hank is a horse, then Hank is not a bird. So, if Hank is a horse, then Hank is a horse and not a bird. (H: Hank is a horse; B: Hank is a bird)

This argument may seem a bit odd, but it is plainly valid. Its form is as follows:

33. $H \rightarrow \sim B \quad \therefore H \rightarrow (H \cdot \sim B)$

Unfortunately, we cannot prove that the argument is valid using only the 18 rules we have in hand so far.[4] In fact, to make our system of statement logic complete, we need to add one further element, a rule called "conditional proof" (CP for short). Without this rule (or some equivalent addition to our system), we would be unable to construct proofs for many valid arguments. CP also greatly simplifies many proofs that in principle could be done without it.

The basic idea behind CP is that *we can prove a conditional true by assuming that its antecedent is true and showing that its consequent can be derived from this assumption* (together with whatever premises are available). For example, take argument (33). We have $H \rightarrow \sim B$ as a premise. We need to show that the premise validly implies the conclusion, which is a conditional statement: $H \rightarrow (H \cdot \sim B)$. We assume H, the antecedent of the conclusion. Now, from the assumption H and the premise $H \rightarrow \sim B$, we can derive $\sim B$ by *modus ponens*. From $\sim B$ and the assumption H, we can obtain $H \cdot \sim B$ by conjunction. This shows that the antecedent, H, of the conditional conclusion leads logically to its consequent, $H \cdot \sim B$, given the premise. Therefore, the argument is valid.

Now, we need to formalize this intuitive proof technique. This means we need a way to include assumptions in our proofs, bearing in mind that an assumption is not a premise. In fact, since conditionals are hypothetical, the antecedent of a conditional may be false (and may be admitted to be false by the arguer) even though the conditional itself is true. So, we need a way of using assumptions temporarily—a way that keeps it clear that we are not treating them as premises. As an example, the formal proof of argument (33) would look like this:

1. $H \rightarrow \sim B$	$\therefore H \rightarrow (H \cdot \sim B)$
2. H	Assume
3. $\sim B$	1, 2, MP
4. $H \cdot \sim B$	2, 3, Conj
5. $H \rightarrow (H \cdot \sim B)$	2–4, CP

The word "Assume" indicates the special status of H. The box indicates the *scope* of the assumption (i.e., the part of the proof in which the assumption is made). The steps from line (2) to line (4) do not prove that $H \cdot \sim B$ follows from the argument's premise. (They *would* prove this if H were a premise and not a mere assumption.) Rather, lines (2) through (4) show only that $H \cdot \sim B$ is true *on the assumption that* H is true. We box in the steps and enter line (5) to make it clear that only a conditional conclusion has been established. The annotation of line (5) mentions the steps falling within the scope of the assumption, as well as the type of proof used (CP). It is crucial to note that line (5) follows logically from the premise of the argument, namely, $H \rightarrow \sim B$. We haven't added a premise to the argument in line (2). We have merely introduced a temporary assumption for the purpose of proving that the *conditional* conclusion follows from the premise.

Using lowercase letters as statement variables, we can make a diagram of conditional proof as follows:

Premises

p	Assume
.	
.	
.	
q	
$p \rightarrow q$	CP

The vertical dots here stand for inferences from the premises and the assumption. In the typical case, $(p \rightarrow q)$ is the conclusion of the argument, though as we will see, this is not necessarily the case.

Let's consider another example:

34. If most Americans favor gun control, then if lobbies block gun control proposals, democracy is hindered. If most Americans favor gun control, then lobbies do block gun control proposals. Therefore, if most Americans favor gun control, democracy is hindered. (M: Most Americans favor gun control; L: Lobbies block gun control proposals; D: Democracy is hindered)

1. $M \rightarrow (L \rightarrow D)$	
2. $M \rightarrow L$	$\therefore M \rightarrow D$
3. M	Assume
4. $L \rightarrow D$	1, 3, MP
5. L	2, 3, MP
6. D	4, 5, MP
7. $M \rightarrow D$	3–6, CP

Notice that it would be a mistake to suppose that the statements within the box have been shown to follow from the premises alone. We box in the statements precisely to remind ourselves of their tentative status, dependent as they are on the assumption in line (3). We stop making our assumption at line (7). And our proof shows that line (7) follows logically from the premises—that is, lines (1) and (2).

When you are making an assumption for the purpose of conditional proof, always select the *antecedent* of the conditional statement that you are trying to obtain. CP is often useful when the conclusion of an argument is a conditional statement. So, we can state the following rule of thumb:

Rule of Thumb 9: If the conclusion of an argument is a conditional statement, use CP.

For instance, consider the following symbolic argument:

35. ~S → W, ~R → U, (U v W) → T ∴ ~(S • R) → (T v Z)

Because the conclusion of this argument is a conditional statement, CP is a good method to try. And we should assume the *antecedent* of the conclusion, ~(*S* • *R*). Accordingly, the proof looks like this:

1. ~S → W	
2. ~R → U	
3. (U v W) → T	∴ ~(S • R) → (T v Z)
4. ~(S • R)	Assume
5. ~S v ~R	4, DeM
6. W v U	5, 1, 2, CD
7. U v W	6, Com
8. T	3, 7, MP
9. T v Z	8, Add
10. ~(S • R) → (T v Z)	4–9, CP

Again, we box in the lines of the proof that fall within the *scope* of the assumption (the part of the proof in which the assumption is made). These lines tell us that if we have ~(*S* • *R*), then we can obtain *T* v Z. The boxed-in steps are *hypothetical* in nature, for they depend on the assumption in line (4). We stop making our assumption at line (10). And our proof shows that line (10) follows validly from the premises—that is, lines (1), (2), and (3).

So far, we have considered cases in which only one assumption is introduced. But sometimes it is helpful to introduce more than one assumption—for example, when you are trying to prove a conditional whose *consequent* is a conditional. Here is an example:

36. If space travelers from another galaxy visit Earth, then aliens will rule us if our technology is inferior. But if our technology is inferior and aliens will rule us, then our liberty will decrease. So, if space travelers from another galaxy visit Earth, then our liberty will decrease if our technology is inferior. (S: Space travelers from another galaxy visit Earth; A: Aliens will rule us; T: Our technology is inferior; L: Our liberty will decrease)

We symbolize the argument and begin a conditional proof in line (3).

1. S → (T → A)	
2. (T • A) → L	∴ S → (T → L)
3. S	Assume
4. T → A	1, 3, MP

Having derived line (4), we could turn our attention to premise (2), applying commutation, exportation, and so on, but with CP another strategy is possible. Note that the conclusion, $S \rightarrow (T \rightarrow L)$, is a conditional with another conditional (namely, $T \rightarrow L$) as its consequent. Thus, we can usefully introduce a second assumption (again, the antecedent of a conditional), as follows:

1. $S \rightarrow (T \rightarrow A)$	
2. $(T \cdot A) \rightarrow L$	$\therefore\ S \rightarrow (T \rightarrow L)$
3. S	Assume
4. $T \rightarrow A$	1, 3, MP
5. T	Assume
6. A	4, 5, MP
7. $T \cdot A$	5, 6, Conj
8. L	2, 7, MP
9. $T \rightarrow L$	5–8, CP

Now, at this point, we have shown that if T, then L, for by assuming T, we were able to obtain L. But all of this occurs within the scope of our first assumption (i.e., S), and *a proof is always incomplete as long as we are still making an assumption*. Furthermore, we have not yet reached the conclusion of the argument, so we need one additional step:

1. $S \rightarrow (T \rightarrow A)$	
2. $(T \cdot A) \rightarrow L$	$\therefore\ S \rightarrow (T \rightarrow L)$
3. S	Assume
4. $T \rightarrow A$	1, 3, MP
5. T	Assume
6. A	4, 5, MP
7. $T \cdot A$	5, 6, Conj
8. L	2, 7, MP
9. $T \rightarrow L$	5–8, CP
10. $S \rightarrow (T \rightarrow L)$	3–9, CP

Lines (3) through (9) indicate that if we have S, we can obtain $T \rightarrow L$. In other words, the proof shows that line (10) follows logically from the premises—that is, lines (1) and (2). So, the argument is valid.

Here is the place to issue two important warnings: First, because the statements within the boxes are dependent on assumptions, we cannot make use of boxed-in statements in later parts of a proof. For example, in the previous proof, it may appear that we could write L on line (9) by applying *modus ponens* to lines (7) and (2), but line (7) is available only because of the assumption in line (5). And the box indicates that we *discharged* (i.e., ceased to make) that assumption when we got to line (9). So, we cannot make use of line (7) in subsequent parts of the proof. In general, boxed-in lines cannot be used to justify later steps in a proof, for the boxes indicate that we have ceased to make the assumption in question. Second, no proof involving CP is complete until all assumptions are discharged.

It should be noted that CP is sometimes useful even when the conclusion of the argument is not a conditional. Here is an example:

37. If God stops people from performing acts that cause unnecessary suffering, then *either* God denies creatures a choice between good and evil, *or* God can cause the free acts of his creatures. If God can cause the free acts of his creatures, then the concept of free will is empty. The concept of free will is not empty. So, either God does not stop people from performing acts that cause unnecessary suffering, or else God denies creatures a real choice between good and evil. (S: God stops people from performing acts that cause unnecessary suffering; G: God denies creatures a choice between good and evil; F: God can cause the free acts of his creatures; W: The concept of free will is empty)

We symbolize the argument and begin a conditional proof in line (4). This makes sense if one realizes that the conclusion, $\sim S \vee G$, is logically equivalent to $S \rightarrow G$.

1. S → (G ∨ F)	
2. F → W	
3. ~W	∴ ~S ∨ G
4. S	Assume
5. G ∨ F	1, 4, MP
6. ~F	2, 3, MT
7. G	5, 6, DS
8. S → G	4–7, CP
9. ~S ∨ G	8, MI

Note that *by CP we always obtain a conditional*, and this case is no exception. Lines (4) through (7) establish $S \rightarrow G$. We then apply MI to obtain the conclusion of the argument.

CP can be used when the conclusion of an argument is a biconditional. For example:

38. (B ∨ A) → C, A → ~C, ~A → B ∴ B ↔ C

The basic strategy is to prove two conditionals, conjoin them, and then use ME:

1. (B ∨ A) → C	
2. A → ~C	
3. ~A → B	∴ B ↔ C
4. B	Assume
5. B ∨ A	4, Add
6. C	1, 5, MP
7. B → C	4–6, CP

8. C	Assume
9. ~~C	8, DN
10. ~A	2, 9, MT
11. B	3,10, MP
12. C → B	8–11, CP
13. (B → C) • (C → B)	7, 12, Conj
14. B ↔ C	13, ME

Note that although two assumptions are made in this proof, neither falls within the scope of the other. So, at line (13), we are free to conjoin lines (7) and (12).

It is possible to construct a direct proof for the previous argument, a **direct proof** being one that makes no use of assumptions. (Try it!) The direct proof is slightly longer than the one just presented, but more important, the direct proof is less intuitive, as it involves the use of MI. We have noted more than once that MI is not an intuitive rule when applied to some English conditionals. Accordingly, it is reassuring that we can often construct a conditional proof without MI when a direct proof would require an application of MI.

Conditional proof renders our system of statement logic complete. Whatever can be proved valid through the truth tables can be proved valid using our 8 implicational rules, 10 equivalence rules, and CP. It is interesting to note that some systems achieve completeness in a different way, by adding the rule of *absorption*, which countenances inferences from $p \rightarrow q$ to $p \rightarrow (p \cdot q)$. However, CP tends to make proofs both shorter and more intuitive than does absorption.[5]

Note: As you complete the following exercises, it may be helpful to refer to the summary of rules of thumb that appears on page 349.

Exercise 8.4

Part A: Conditional Proofs Use CP to show that each of the following symbolic arguments is valid.

* 1. Z → (~Y → X), Z → ~Y ∴ Z → X

2. P → Q ∴ P → (Q ∨ R)

3. (F ∨ ~G) → ~L ∴ L → G

* 4. A → B, A → C ∴ A → (B • C)

5. (H ∨ E) → K ∴ E → K

6. (B → ~C) → D ∴ B → (~C → D)

* 7. P ∴ (P → Q) → Q

8. S ∴ ~(S • R) → ~R

9. (G → H) → J ∴ H → J

* 10. C → (~D → E), (D → ~D) → (E → G) ∴ C → (~D → G)

11. H → (J • K), ~L → (J • M) ∴ (~L ∨ H) → J

12. $\sim X \vee (O \cdot W),\ (X \to O) \to (W \to X)\ \therefore W \to X$

*13. $(A \vee N) \to \sim S,\ M \to [N \to (S \cdot T)]\ \therefore \sim(\sim M \vee \sim N) \to (S \cdot \sim A)$

14. $\sim P \vee (Q \cdot \sim R)\ \therefore (R \vee R) \to \sim P$

15. $(S \vee T) \leftrightarrow \sim E,\ S \to (F \cdot \sim G),\ A \to W,\ T \to \sim W\ \therefore (\sim E \cdot A) \to \sim G$

*16. $A \to (B \to C)\ \therefore (A \to B) \to (A \to C)$

17. $(G \cdot P) \to K,\ E \to Z,\ \sim P \to \sim Z,\ G \to (E \vee L)\ \therefore (G \cdot \sim L) \to K$

18. $S \to (\sim T \to U),\ \sim T \to (U \to O)\ \therefore \sim S \vee [(T \to \sim T) \to O]$

*19. $A \to (B \cdot C),\ B \to D,\ C \to \sim D\ \therefore A \to X$

20. $B \to [(E \cdot \sim G) \to M],\ \sim(\sim E \vee G) \to (M \to R)$
$\therefore B \to [\sim(\sim G \to \sim E) \to R]$

21. $P \to (Q \to R)\ \therefore Q \to (P \to R)$

22. $Q \to R\ \therefore (P \vee Q) \to (P \vee R)$

23. $A \leftrightarrow B\ \therefore \sim B \leftrightarrow \sim A$

24. $C \leftrightarrow D,\ D \leftrightarrow \sim E\ \therefore C \leftrightarrow \sim E$

25. $\sim A\ \therefore [(A \cdot B) \vee (C \cdot D)] \leftrightarrow [(A \vee C) \cdot (A \vee D)]$

Part B: English Arguments Symbolize the following arguments, using the schemes of abbreviation provided. Then use CP to show that the arguments are valid.

* 1. If Jones doesn't vote, then he shouldn't vote. For after all, if Jones doesn't vote, then either he lacks intelligence or he lacks a proper value system. And Jones shouldn't vote if he lacks intelligence. Furthermore, Jones shouldn't vote if he lacks a proper value system. (V: Jones does vote; I: Jones has intelligence; P: Jones has a proper value system; S: Jones should vote)

2. Euthanasia is wrong if either the patient prefers to go on living or she still maintains her higher faculties. Therefore, if the patient still maintains her higher faculties, then euthanasia is wrong. (P: The patient prefers to go on living; F: The patient maintains her higher faculties; E: Euthanasia is wrong)

3. If we should forgive our enemies, then it is wrong to punish criminals. For if we should forgive our enemies, then we should forget the offense and behave as if the offense never occurred. And we should punish criminals if and only if we should not behave as if the offense never occurred. Furthermore, it is wrong to punish criminals if and only if we should not punish criminals. (F: We should forgive our enemies; W: It is wrong to punish criminals; O: We should forget the offense; B: We should behave as if the offense never occurred; S: We should punish criminals)

4. If God believes on Monday that I'll tell a lie on Tuesday, then either I have the power to make one of God's past beliefs false, or I cannot refrain from lying on Tuesday. I do not have the power to make one of God's past beliefs

false if either God is infallible or the past is unalterable. The past is unalterable. It follows that if God believes on Monday that I'll tell a lie on Tuesday, then I cannot refrain from lying on Tuesday. (B: God believes on Monday that I'll tell a lie on Tuesday; F: I have the power to make one of God's past beliefs false; R: I can refrain from lying on Tuesday; I: God is infallible; P: The past is unalterable)

5. If humans lack free will, then there is no moral responsibility. Materialism is true if and only if only matter exists. Assuming that only matter exists, every event is the result of past states of the world plus the operation of natural laws. Now, if every event is the result of past states of the world plus the operation of natural laws, then human acts are under human control only if either humans have control over the past or humans have control over the natural laws. Humans do not have control over the past, and they do not have control over the natural laws. Finally, if human acts are not under human control, then humans do not have free will. We may conclude that if materialism is true, then there is no moral responsibility. (F: Humans have free will; R: There is moral responsibility; M: Materialism is true; O: Only matter exists; E: Every event is the result of past states of the world plus the operation of natural laws; C: Human acts are under human control; P: Humans have control over the past; N: Humans have control over the natural laws)

Part C: Valid or Invalid? Symbolize the following arguments. If an argument is invalid, prove this by means of an abbreviated truth table. If an argument is valid, construct a proof to demonstrate its validity.

1. If either moral judgments are products of biological causes or moral judgments are not based on empirical evidence, then morality is not objective. But if moral judgments are not products of biological causes, then moral judgments are not based on empirical evidence. Hence, morality is not objective. (M: Moral judgments are products of biological causes; E: Moral judgments are based on empirical evidence; O: Morality is objective)

2. It is false that if we continue to use gasoline, then the air will not be polluted. Either we do not continue to use gasoline or we use solar power. If we continue to use gasoline and air-pollution control devices are perfected, then the air will not be polluted. Therefore, we use solar power if and only if air-pollution control devices are perfected. (G: We continue to use gasoline; A: The air will be polluted; S: We use solar power; P: Air-pollution control devices are perfected)

3. Given that Henri Rousseau's *The Dream* is pornographic if and only if Rousseau painted it with the intention of inciting lust in the viewers, Rousseau's *The Dream* is not pornographic. For Rousseau painted it with the intention of inciting lust in the viewers only if every nude painting is painted with the intention of inciting lust in the viewers. And the latter suggestion is wildly false! (P: Henri Rousseau's *The Dream* is pornographic; L: Rousseau painted *The Dream* with the intention of inciting lust in the

viewers; N: Every nude painting is painted with the intention of inciting lust in the viewers)

4. If Boethius is morally virtuous, then he achieves heaven. But if he isn't morally virtuous, then his longings are satisfied. On the other hand, if Boethius doesn't achieve heaven, then his longings are not satisfied. So, Boethius's longings are satisfied. (M: Boethius is morally virtuous; H: Boethius achieves heaven; L: Boethius's longings are satisfied)
5. Either God has a reason for his commands, or morality is ultimately arbitrary. If God has a reason for his commands, then reasons that are independent of God's will make actions right. Consequently, reasons that are independent of God's will make actions right provided that morality is not ultimately arbitrary. (R: God has a reason for his commands; M: Morality is ultimately arbitrary; I: Reasons that are independent of God's will make actions right)

8.5 Reductio ad Absurdum

Although our system of statement logic is already complete, we will add one more rule that simplifies proofs in many cases, namely, *reductio ad absurdum* (RAA for short). The basic principle behind RAA is this: *Whatever implies a contradiction is false*. Using the italicized, lowercase letters p and q as statement variables (which can stand for any statement), we can see that RAA is closely related to *modus tollens*. Suppose we know that a given statement $\sim p$ implies a contradiction:

39. $\sim p \rightarrow (q \cdot \sim q)$

Now, we know that contradictions are false. So, we also know this:

40. $\sim(q \cdot \sim q)$

But, then if we apply *modus tollens* to (39) and (40), we get $\sim\sim p$ and, hence, p by DN. This is the essential logic underlying *reductio ad absurdum*. Since $\sim p$ leads to (or "reduces" to) a logical absurdity (i.e., a contradiction), $\sim p$ must be false, and hence p is true. Now, in practice, the contradiction does not usually follow from a single statement all by itself. Rather, the contradiction usually follows from the premises of the argument (which are taken as true for the purpose of establishing validity) *together with* the temporary assumption, $\sim p$, where p is the conclusion of the argument.

Look at it this way. Suppose we have three statements that together imply a contradiction. For instance:

$\sim A \rightarrow (B \cdot \sim C)$
$B \rightarrow C$
$\sim A$

Using MP, Simp, and Conj, one can derive C • ~C from these statements in only a few steps. Because these statements imply a contradiction, we know that at least one of them is false. Now, given that the first two statements are true, we can conclude that ~A is false and hence that A is true. This reasoning shows the following argument to be valid:

41. ~A → (B • ~C), B → C ∴ A

The formal proof runs as follows:

1. ~A → (B • ~C)	
2. B → C	∴ A
3. ~A	Assume
4. B • ~C	1, 3, MP
5. B	4, Simp
6. C	2, 5, MP
7. ~C	4, Simp
8. C • ~C	6, 7, Conj
9. A	3–8, RAA

For the purpose of establishing the *validity* of an argument, the truth of the premises is a given. So, since the premises, together with ~A, imply a contradiction, we may conclude that ~A is false, and hence that A is true. As with CP, we box in the lines that fall within the scope of the assumption and add line (9) to indicate that A follows not from our assumption but from the premises of the argument. The annotation for line (9) mentions the lines falling within the scope of the assumption and adds "RAA" for *reductio ad absurdum*.

When the conclusion of an argument is the *negation* of a statement, (e.g., ~B), your assumption line should usually be the statement itself (in this case, B) rather than a double-negation. This procedure will usually save some steps. For example, consider the following proof:

1. B ↔ ~A	
2. ~A → ~C	
3. C v D	
4. ~C → ~D	∴ ~B
5. B	Assume
6. (B → ~A) • (~A → B)	1, ME
7. B → ~A	6, Simp
8. ~A	5, 7, MP
9. ~C	8, 2, MP
10. D	3, 9, DS
11. ~D	4, 9, MP
12. D • ~D	10, 11, Conj
13. ~B	5–12, RAA

Note that in line (5), we assume B rather than ~~B. It wouldn't be a logical error to assume ~~B, but it would add an unnecessary step. (We'd have to apply DN to drop the double-negation prior to performing a *modus ponens* step.)

Thus, a proof involving RAA may proceed in two ways. When we are trying to prove a negation, we obtain our assumption line simply by dropping the tilde. When we are trying to prove a statement that is not a negation, we obtain our assumption line by *adding* a tilde. Using lowercase letters as statement variables, we can make a diagram of these two forms of RAA as follows:

To Prove a Negation: ~*p*

Premises	
p	Assume
.	
.	
.	
(*q* • ~*q*)	
~*p*	RAA

To Prove a Statement That Is Not a Negation: *p*

Premises	
~*p*	Assume
.	
.	
.	
(*q* • ~*q*)	
p	RAA

The procedure is essentially the same in both cases: We show that a statement (together with the premises) implies a contradiction and conclude that the statement is false. Note: As with CP, no proof involving RAA is complete until all assumptions have been discharged.

When should one use RAA? There is usually no way to know for sure, apart from experiment, whether RAA will prove useful, but here are some points to keep in mind. First, RAA will always work (assuming, of course, that the argument is valid), but RAA may unnecessarily complicate a proof. Second, when direct proof seems difficult or impossible and the conclusion of the argument is not a conditional, try RAA. (If the conclusion is a conditional, CP is usually preferable to RAA.) Consider an example:

42. (F ∨ ~F) → G ∴ G

Applying MI to the premise, we get ~(*F* ∨ ~*F*) ∨ G. By DeM we can then obtain (~*F* • ~~*F*) ∨ G. Com will give us G ∨ (~*F* • ~~*F*). And Dist will yield (G ∨ ~*F*) • (G ∨ ~~*F*). Now we can simplify to obtain G ∨ ~*F* as well as G ∨ ~~*F*. But where do we go from here? Maybe it would help to have an assumption to work with. And since the conclusion is not a conditional, let's try RAA:

1. (F ∨ ~F) → G	∴ G
2. ~G	Assume
3. ~(F ∨ ~F)	1, 2, MT
4. ~F • ~~F	3, DeM
5. G	2–4, RAA

In this case, RAA makes the proof short and easy. Let us now add a tenth rule of thumb:

> **Rule of Thumb 10:** If direct proof is difficult and the conclusion of the argument is not a conditional, try RAA.

RAA and CP are closely related from a theoretical standpoint. For example, we can always use CP whenever we use RAA. To illustrate, consider the following proofs:

1. ~P → (Q • R)	
2. R → ~Q	∴ P
3. ~P	Assume
4. Q • R	1, 3, MP
5. R	4, Simp
6. ~Q	2, 5, MP
7. Q	4, Simp
8. Q ∨ P	7, Add
9. P	6, 8, DS
10. ~P → P	3–9, CP
11. ~~P ∨ P	10, MI
12. P ∨ P	11, DN
13. P	12, Re

1. ~P → (Q • R)	
2. R → ~Q	∴ P
3. ~P	Assume
4. Q • R	1, 3, MP
5. R	4, Simp
6. ~Q	2, 5, MP
7. Q	4, Simp
8. Q • ~Q	6, 7, Conj
9. P	3–8, RAA

Note that the CP proof is exactly like the RAA proof down to line (7). The remaining steps are characteristic of ones we could employ *whenever we have derived a contradiction from an assumption*. So, there is a close theoretical relationship between CP and RAA. RAA proofs, however, will typically be shorter *except when* the conclusion of the argument is a conditional.

It is also worth noting that in principle, we could dispense with CP and use RAA to complete our system of statement logic, for RAA works whenever CP works. To illustrate, consider the following RAA proof for an argument having a conditional as its conclusion:

Summary of Rules of Thumb for Constructing Proofs

1. It usually helps to work backward. So, start by looking at the conclusion. Try to find the conclusion (or elements thereof) in the premises.
2. Apply the inference rules to break down the premises.
3. If the conclusion contains a statement letter that does not appear in the premises, use the rule of addition.
4. It is often useful to consider logically equivalent forms of the conclusion.
5. Both conjunction and addition can lead to useful applications of De Morgan's laws.
6. Material implication can lead to useful applications of distribution.
7. Distribution can lead to useful applications of simplification.
8. Addition can lead to useful applications of material implication.
9. If the conclusion of an argument is a conditional statement, use CP.
10. If direct proof is difficult and the conclusion of the argument is not a conditional, try RAA.

1. Z → (~Y → X)	
2. Z → ~Y	∴ Z→ X
3. ~(Z → X)	Assume
4. ~(~Z ∨ X)	3, MI
5. ~~Z • ~X	4, DeM
6. ~~Z	5, Simp
7. Z	6, DN
8. ~Y	2, 7, MP
9. ~Y → X	1, 7, MP
10. X	8, 9, MP
11. ~X	5, Simp
12. X • ~X	10, 11, Conj
13. Z → X	3–12, RAA

While from a purely theoretical standpoint we do not need both CP *and* RAA, both rules are intuitive and both are quite useful. Thus, it is important to be able to employ both of them. In this regard, we can note that it is easy to construct a *seven*-step CP proof for the previous argument, a fact that underscores rule of thumb 9: If the conclusion of an argument is a conditional, use CP.

It is possible to combine RAA and CP. Here is an example:

1.	~(S • ~R) v (S → T)	∴ S → (R v T)
2.	S	Assume (for CP)
3.	~(R v T)	Assume (for RAA)
4.	~R • ~T	3, DeM
5.	~R	4, Simp
6.	S • ~R	2, 5, Conj
7.	~~(S • ~R)	6, DN
8.	S → T	1, 7, DS
9.	T	2, 8, MP
10.	~T	4, Simp
11.	T • ~T	9,10, Conj
12.	R v T	3–11, RAA
13.	S → (R v T)	2–12, CP

In line (2), we begin a conditional proof. Having begun a CP proof, we need to obtain the consequent of the conditional in question, namely, $R \vee T$. If we assume $\sim(R \vee T)$ and derive a contradiction, we will have shown that $R \vee T$ must be true given that S is true. The preceding proof spells out the details. Note that in this case, an RAA proof falls within the scope of a CP proof.

As we have seen, when using RAA, one typically derives a contradiction from the assumption that the *conclusion* of the argument is false. But other assumptions can be useful. Here's an example:

1.	L → H	
2.	L → ~H	
3.	~L → (S v R)	
4.	~R	∴ S
5.	L	Assume
6.	H	1, 5, MP
7.	~H	2, 5, MP
8.	H • ~H	6, 7, Conj
9.	~L	5–8, RAA
10.	S v R	9, 3, MP
11.	S	10, 4, DS

Why assume L at line (5)? This assumption makes sense for two reasons. First, if we can obtain $\sim L$, then obviously we can derive S from premises (3) and (4). Second, given premises (1) and (2), if we assume L, we can easily derive a contradiction.

Note: as with CP, because the statements within the boxes of an RAA proof are dependent on one or more assumptions, we cannot make use of boxed-in statements in *later* parts of a proof. For example, could we enter ~L • H at line (10) in the above proof, using "6, 9, Conj" as our annotation? No. For at that point in the proof, line (6) is off limits—we obtained *H* by making an assumption, and (as the box indicates) we stopped making that assumption when we got to line (9).

As you complete the exercises for this section, you may find the summary of rules of thumb helpful.

Exercise 8.5

Part A: Proofs Construct proofs to show that the following symbolic arguments are valid. Use RAA but not CP.

* 1. A → B ∴ ~(A • ~B)

2. P → Q, ~P → J, ~Q → ~J ∴ Q

3. F → G, F ∨ G ∴ G

* 4. (H ∨ R) • (H ∨ ~R) ∴ H

5. (M → L) → M ∴ M

6. ~P ↔ Q, ~(Q ∨ R), (P • ~R) → S ∴ S

* 7. Z → (X ∨ Y), X → ~W, Y → ~W, ~W → ~Z ∴ ~Z

8. E ∨ T, T → (B • H), (B ∨ E) → K ∴ K

9. (O ∨ N) → (O • N) ∴ N ↔ O

* 10. ~A • ~B ∴ A ↔ B

11. ~W ∨ (Z → Y), ~X → (W ∨ Y), W → Z ∴ Y ∨ X

12. ~P → (R • S), ~Q → (R • T), ~(S ∨ T) ∴ P • Q

* 13. D → ~(A ∨ B), ~C → D ∴ A → C

14. E ∴ (E • H) ∨ (E • ~H)

15. ~Q → (L → F), Q → ~A, F → B, L ∴ ~A ∨ B

* 16. W → (X ∨ G), G → M, ~M ∴ ~W ∨ X

17. (~H ∨ K) • (~H ∨ L), ~N → H, ~N → (~L ∨ ~K), P ↔ N ∴ S → P

18. C → (D → H), D • ~H, H ∨ T ∴ ~C • T

* 19. ~S → (T • U), ~R → ~(T ∨ U), (T ↔ U) → (~~S • R) ∴ R • S

20. (A → B) → (C → A) ∴ C → A

21. (S ∨ T) ∨ (V ∨ W) ∴ (V ∨ T) ∨ (S ∨ W)

22. S → T ∴ (U ∨ S) → (U ∨ T)

23. X → Y ∴ (Y ∨ X) → Y

24. R • P, R → (S ∨ Q), ~(Q • P) ∴ S

25. K, D → E, D → F, D → G, J → ~K, E → H, H → [I → (~J → ~G)], F → (K → I) ∴ ~D

Part B: Valid or Invalid? For each of the following pairs of arguments, one is valid and one is invalid. Use an abbreviated truth table to determine which argument is invalid. Then construct a proof to show that the other member of the pair is valid, using either RAA or CP.

* 1. $(F \to G) \to H \therefore F \to (G \to H)$

* 2. $F \to (G \to H) \therefore (F \to G) \to H$

3. $\sim L \to L, \sim L \leftrightarrow N \therefore \sim N$

4. $(E \bullet F) \to G \therefore F \to G$

5. $(\sim D \vee H) \bullet (\sim D \vee \sim P), \sim D \to S, (H \bullet P) \to \sim U \therefore S \vee \sim U$

6. $\sim(S \to R) \therefore S \bullet \sim R$

7. $(Z \vee Y) \bullet (Z \vee W) \therefore Z \bullet (Y \vee W)$

8. $P \to \sim Q \therefore Q \to \sim P$

9. $\sim S \to (F \to L), F \to (L \to P) \therefore \sim S \to (F \to P)$

10. $A \to (B \vee C) \therefore (A \to B) \bullet (\sim A \vee C)$

Part C: English Arguments Symbolize the following arguments using the schemes of abbreviation provided. Then construct proofs to show that the arguments are valid. Use only RAA.

* 1. If the rate of literacy has declined, then either TV or parental neglect is the cause. If TV is the cause, then we can't increase literacy unless we can get rid of TV. If parental neglect is the cause, then we can't increase literacy unless we are willing to support early childhood education with our tax dollars. The rate of literacy has declined, but we can't get rid of TV and we certainly aren't willing to support early childhood education with our tax dollars. So, we can't increase literacy. (R: The rate of literacy has declined; T: TV is the cause of the decline in the rate of literacy; P: Parental neglect is the cause of the decline in the rate of literacy; L: We can increase literacy; C: We can get rid of TV; W: We are willing to support early childhood education with our tax dollars)

2. Either vegetarians are misguided, or factory farming is cruel and the grain fed to animals could save thousands of starving people. Vegetarians are misguided only if feeding grain to animals is an efficient way to make protein. And if the grain fed to animals could save thousands of starving people, then American consumers are insensitive if they insist on eating meat at the current rate. American consumers insist on eating meat at the current rate. Therefore, either American consumers are insensitive, or feeding grain to animals is an efficient way to make protein. (V: Vegetarians are misguided; F: Factory farming is cruel; G: The grain fed to animals could save thousands of starving people; E: Feeding grain to animals is an efficient way to make pro-

tein; A: American consumers are insensitive; M: American consumers insist on eating meat at the current rate)

3. We should maximize the general welfare if and only if utilitarianism is true. If we should maximize the general welfare, we should promote the greatest sum of pleasure. If we should promote the greatest sum of pleasure, then we are morally obligated to increase the size of the population provided that we can increase the size of the population without reducing the standard of living. We can increase the size of the population without reducing the standard of living, but we are not morally obligated to increase the size of the population if either increasing the size of the population will destroy the environment or no individual experiences the sum of pleasure. Obviously, no individual experiences the sum of pleasure. It follows that utilitarianism is not true. (W: We should maximize the general welfare; U: Utilitarianism is true; P: We should promote the greatest sum of pleasure; M: We are morally obligated to increase the size of the population; R: We can increase the size of the population without reducing the standard of living; E: Increasing the size of the population will destroy the environment; N: No individual experiences the sum of pleasure)

4. According to some Hindu traditions, reincarnation is true, but reality is undifferentiated being. However, it is not the case that *both* reincarnation is true *and* reality is undifferentiated being. For reincarnation is true if and only if a person's soul transfers to another body at death. But if a person's soul transfers to another body at death, then each individual soul is real and each individual soul differs from all other souls. But if reality is undifferentiated being, then all apparent differences are illusory. And if each individual soul is real, then souls are not illusory. However, if each individual soul differs from all other souls and souls are not illusory, then not all apparent differences are illusory. (R: Reincarnation is true; U: Reality is undifferentiated being; T: A person's soul transfers to another body at death; E: Each individual soul is real; D: Each individual soul differs from all other souls; S: Souls are illusory; A: All apparent differences are illusory)

5. Some hold the view that while contradictions *could* be true, we happen to know that they are always false. This view is mistaken. For if contradictions could be true, then if the evidence for some statements is counterbalanced by equally strong evidence for their negations, some contradictions are true for all we know. Now, if there are areas of controversy among scholars, then the evidence for some statements is counterbalanced by equally strong evidence for their negations. And it almost goes without saying that there are areas of controversy among scholars. Finally, some contradictions are true for all we know if and only if we do not know that contradictions are always false. (C: Contradictions could be true; K: We know that contradictions are always false; E: The evidence for some statements is counterbalanced by equally strong evidence for their negations; S: Some contradictions are true for all we know; A: There are areas of controversy among scholars)

Part D: Valid or Invalid? Symbolize the following arguments using the schemes of abbreviation provided. If an argument is invalid, demonstrate this by means of an abbreviated truth table. (Only one of the arguments is invalid.) If an argument is valid, demonstrate this by constructing a proof. You can use CP, RAA, or direct proof.

1. If Smith works hard, then he gets elected. But if he doesn't work hard, then he is happy. Moreover, if he doesn't get elected, then he isn't happy. We may infer that Smith gets elected. (W: Smith works hard; E: Smith gets elected; H: Smith is happy)
2. If either mathematical laws are due to arbitrary linguistic conventions or mathematical laws are not based on empirical evidence, then math is merely a game played with symbols. If mathematical laws are not based on empirical evidence, then they are not due to arbitrary linguistic conventions. So, math is merely a game played with symbols. (M: Mathematical laws are due to arbitrary linguistic conventions; E: Mathematical laws are based on empirical evidence; G: Math is merely a game played with symbols)
3. God is not outside of time if time is real. For, as St. Thomas Aquinas pointed out, if God is outside of time, then God sees all of time (past, present, and future) at a glance. But if God sees all of time at a glance, then all of time (past, present, and future) already exists. Now, if all of time already exists, then the future already exists. However, if the future already exists, then I have already committed sins that I will commit in the future. But if time is real, I have emphatically *not* already committed sins that I will commit in the future. (O: God is outside of time; S: God sees all of time at a glance; A: All of time already exists; F: The future already exists; I: I have already committed sins that I will commit in the future; T: Time is real)
4. Television has destroyed the moral fiber of our country if it has both stifled creativity and substantially interfered with communication between children and parents. Of course, television has not destroyed the moral fiber of our country, assuming that our country still has moral fiber. However, one must admit that television has substantially interfered with communication between children and parents. Furthermore, the statement "Television has stifled creativity if and only if television is a good thing" is false. It follows that television is a good thing given that our country still has moral fiber. (T: Television has destroyed the moral fiber of our country; S: Television has stifled creativity; C: Television has substantially interfered with communication between children and parents; M: Our country still has moral fiber; G: Television is a good thing)
5. There is life after death if and only if there is a God. For either God exists or only matter exists. And if only matter exists, then when we die our bodies simply decay and we cease to exist permanently. Of course, if we cease to exist permanently, then there is no life after death. But God exists if and only if God is both perfectly good and omnipotent. If God is

omnipotent, God is able to raise humans from the dead. If God is perfectly good, then God wants to raise humans from the dead if resurrection is necessary for their fulfillment. Resurrection is necessary for human fulfillment if most people die with their deepest longings unsatisfied; and as a matter of fact most people do die in that condition. If God is able *and* wants to raise humans from the dead, then there is life after death. (L: There is life after death; G: God exists; M: Only matter exists; D: When we die our bodies simply decay; E: We cease to exist permanently; P: God is perfectly good; O: God is omnipotent; A: God is able to raise humans from the dead; W: God wants to raise humans from the dead; R: Resurrection is necessary for human fulfillment; U: Most people die with their deepest longings unsatisfied)

8.6 Proving Theorems

A **theorem** is a statement that can be proved independently of any premises. The theorems of statement logic are identical with the tautologies of statement logic. (Recall that a *tautology* is a statement that is true in every row of its truth table.) Theorems belong to a class of statements that are true by virtue of their logical form. Many philosophers regard theorems as one type of necessary truth. A **necessary truth** is a truth that cannot be false under any possible circumstances.

Theorems have some rather paradoxical logical properties. For instance, any argument that has a theorem as its conclusion is valid, regardless of the information in the premises. This is so because it is impossible for a theorem to be false, and hence it is impossible for the conclusion of such an argument to be false while the premises are true. Note that this implies that each theorem is validly implied by any other theorem.

To prove a theorem, use either CP or RAA. If the theorem is itself a conditional statement, it is usually best to use CP. Here is an example:

	∴ ~A → [(A ∨ B) → B]
1. ~A	Assume
2. A ∨ B	Assume
3. B	1, 2, DS
4. (A ∨ B) → B	2–3, CP
5. ~A → [(A ∨ B) → B]	1–4, CP

The theorem itself is indicated by the triple-dot symbol. This proof shows that if we have ~A, then if we have A ∨ B, we can derive B. In other words, the proof shows that the statement beside the triple-dot symbol is indeed a theorem: It can be proved without appealing to any premises.

In some cases, RAA is the best approach. Here is a simple example:

	∴ $P \vee {\sim}P$
1. $\sim(P \vee {\sim}P)$	Assume
2. $\sim P \bullet {\sim\sim}P$	1, DeM
3. $P \vee {\sim}P$	1–2, RAA

In other cases, a combination of CP and RAA works best. For instance:

	∴ $[(F \rightarrow G) \rightarrow F] \rightarrow F$
1. $(F \rightarrow G) \rightarrow F$	Assume (for CP)
2. $\sim F$	Assume (for RAA)
3. $\sim(F \rightarrow G)$	1, 2, MT
4. $\sim(\sim F \vee G)$	3, MI
5. $\sim\sim F \bullet {\sim}G$	4, DeM
6. $\sim\sim F$	5, Simp
7. $\sim F \bullet {\sim\sim}F$	2, 6, Conj
8. F	2–7, RAA
9. $[(F \rightarrow G) \rightarrow F] \rightarrow F$	1–8, CP

Sometimes it is necessary to introduce multiple assumptions to prove a theorem. Here is an example:

	∴ $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$
1. $A \rightarrow (B \rightarrow C)$	Assume
2. $A \rightarrow B$	Assume
3. A	Assume
4. B	2, 3, MP
5. $B \rightarrow C$	1, 3, MP
6. C	4, 5, MP
7. $A \rightarrow C$	3–6, CP
8. $(A \rightarrow B) \rightarrow (A \rightarrow C)$	2–7, CP
9. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$	1–8, CP

There is an important connection between valid arguments and theorems. To understand this connection, we first need the concept of a **corresponding conditional.** In the case of an argument with a single premise, one forms the corresponding conditional simply by connecting the premise and conclusion with an arrow. Here is an example:

Argument: $\sim(A \vee {\sim}B)$ ∴ B
Corresponding conditional: $\sim(A \vee {\sim}B) \rightarrow B$

In the case of an argument with multiple premises, forming the corresponding conditional is a two-step process. First, one conjoins the premises—that is, one forms a conjunction of the premises. Second, one connects this conjunction with the conclusion of the argument by means of an arrow. To illustrate:

Argument: $P \rightarrow Q,\ \sim Q\ \therefore\ \sim P$
Conjunction of premises: $(P \rightarrow Q) \bullet \sim Q$
Corresponding conditional: $[(P \rightarrow Q) \bullet \sim Q] \rightarrow \sim P$

Note that in this case, the form of the argument is *modus tollens*. Of course, the argument is valid, and the corresponding conditional is a theorem. This is a relationship that can be counted on for every symbolic argument of statement logic: A symbolic argument is valid if and only if its corresponding conditional is a theorem.

Consider a second example. The argument form is traditionally known as *destructive dilemma*:

Argument: $\sim A \vee \sim B,\ C \rightarrow A,\ D \rightarrow B\ \therefore\ \sim C \vee \sim D$

To form the corresponding conditional, we first make a conjunction out of the premises, like this:

$$(\sim A \vee \sim B) \bullet [(C \rightarrow A) \bullet (D \rightarrow B)]$$

Next, we connect this conjunction to the conclusion of the argument with an arrow, to obtain the corresponding conditional:

$$((\sim A \vee \sim B) \bullet [(C \rightarrow A) \bullet (D \rightarrow B)]) \rightarrow (\sim C \vee \sim D)$$

Now, we can prove that the argument is valid by proving that its corresponding conditional is a theorem:

$\therefore\ ((\sim A \vee \sim B) \bullet [(C \rightarrow A) \bullet (D \rightarrow B)]) \rightarrow (\sim C \vee \sim D)$

1.	$(\sim A \vee \sim B) \bullet [(C \rightarrow A) \bullet (D \rightarrow B)]$	Assume
2.	$\sim A \vee \sim B$	1, Simp
3.	$(C \rightarrow A) \bullet (D \rightarrow B)$	1, Simp
4.	$C \rightarrow A$	3, Simp
5.	$D \rightarrow B$	3, Simp
6.	$\sim(\sim C \vee \sim D)$	Assume
7.	$\sim\sim C \bullet \sim\sim D$	6, DeM
8.	$\sim\sim C$	7, Simp
9.	C	8, DN
10.	A	4, 9, MP
11.	$\sim\sim A$	10, DN
12.	$\sim B$	2, 11, DS
13.	$\sim\sim D$	7, Simp
14.	D	13, DN
15.	B	5, 14, MP
16.	$B \bullet \sim B$	15, 12, Conj
17.	$\sim C \vee \sim D$	6–16, RAA
18.	$((\sim A \vee \sim B) \bullet [(C \rightarrow A) \bullet (D \rightarrow B)]) \rightarrow (\sim C \vee \sim D)$	1–17, CP

The following exercises will provide you with practice in constructing proofs for theorems.

Exercise 8.6

Part A: Theorems Prove the following theorems using either CP or RAA.

* 1. $\sim(P \rightarrow Q) \rightarrow (P \bullet \sim Q)$
2. $\sim(A \bullet \sim A)$
3. $[(S \vee R) \bullet \sim R] \rightarrow S$
* 4. $(X \rightarrow Y) \rightarrow \sim(X \bullet \sim Y)$
5. $(\sim F \bullet \sim G) \rightarrow (F \leftrightarrow G)$
6. $\sim(H \bullet [(H \rightarrow J) \bullet (H \rightarrow \sim J)])$
* 7. $K \rightarrow [(K \rightarrow L) \rightarrow L]$
8. $\sim(M \leftrightarrow \sim M)$
9. $(\sim N \rightarrow O) \vee (N \rightarrow O)$
* 10. $(P \bullet \sim Q) \rightarrow \sim(P \leftrightarrow Q)$
11. $[(\sim B \rightarrow \sim A) \rightarrow A] \rightarrow A$
12. $\sim[(X \leftrightarrow Y) \bullet \sim(X \vee \sim Y)]$
13. $\sim F \rightarrow (F \rightarrow G)$
14. $[\sim H \vee (\sim J \vee K)] \rightarrow [(\sim H \vee J) \rightarrow (\sim H \vee K)]$
15. $[(\sim M \vee M) \rightarrow M] \rightarrow M$
16. $[(P \rightarrow Q) \bullet (R \rightarrow \sim Q)] \rightarrow \sim(P \bullet R)$
17. $D \rightarrow (C \rightarrow D)$
18. $\sim[(E \vee F) \bullet ((E \rightarrow G) \bullet [(F \rightarrow G) \bullet \sim G])]$
19. $(\sim X \rightarrow Y) \vee (X \rightarrow Z)$
20. $[(A \rightarrow B) \vee (A \rightarrow C)] \rightarrow [A \rightarrow (B \vee C)]$

Part B: Challenging Theorems Prove the following theorems using either CP or RAA.

* 1. $(T \rightarrow U) \vee (U \rightarrow T)$
2. $(D \rightarrow E) \rightarrow [(F \rightarrow E) \rightarrow ((D \vee F) \rightarrow E)]$
3. $[(H \rightarrow I) \rightarrow H] \rightarrow H$
* 4. $[P \vee (\sim P \bullet Q)] \leftrightarrow (P \vee Q)$
5. $(R \leftrightarrow S) \rightarrow [((T \rightarrow R) \leftrightarrow (T \rightarrow S)) \bullet ((R \rightarrow T) \leftrightarrow (S \rightarrow T))]$
6. $[(S \vee T) \bullet (Q \vee R)] \rightarrow [((S \bullet Q) \vee (S \bullet R)) \vee ((T \bullet Q) \vee (T \bullet R))]$

* 7. $[((L \cdot M) \vee (L \cdot N)) \vee ((P \cdot M) \vee (P \cdot N))] \rightarrow [(L \vee P) \cdot (M \vee N)]$

8. $[(K \rightarrow J) \cdot (Q \rightarrow R)] \rightarrow [((\sim K \cdot \sim Q) \vee (\sim K \cdot R)) \vee ((J \cdot \sim Q) \vee (J \cdot R))]$

9. $[((\sim E \cdot \sim G) \vee (\sim E \cdot H)) \vee ((F \cdot \sim G) \vee (F \cdot H))] \rightarrow [(E \rightarrow F) \cdot (G \rightarrow H)]$

* 10. $[(A \cdot B) \vee (C \cdot D)] \rightarrow [((A \vee C) \cdot (A \vee D)) \cdot ((B \vee C) \cdot (B \vee D))]$

Part C: Corresponding Conditionals Form the corresponding conditional for each of the following symbolic arguments. Then construct a proof to show that each of the conditionals is a theorem.

* 1. $\sim A \vee \sim B,\ B\ \therefore \sim A$

2. $C \leftrightarrow D,\ C\ \therefore D$

3. $\sim E\ \therefore E \rightarrow F$

4. $G \rightarrow J,\ \sim K \rightarrow \sim H,\ G \vee H\ \therefore J \vee K$

5. $\sim M \vee \sim S,\ \sim L\ \therefore (\sim L \cdot \sim M) \vee (\sim L \cdot \sim S)$

6. $N \cdot O\ \therefore P \rightarrow N$

7. $\sim R,\ Q\ \therefore \sim(Q \rightarrow R)$

8. $\sim S \vee T,\ \sim T \vee U\ \therefore U \vee \sim S$

9. $\sim(W \rightarrow X),\ Z \rightarrow X\ \therefore \sim Z$

10. $A \leftrightarrow \sim A\ \therefore B$

Notes

1. The relevant work is Gerhard Gentzen, "Untersuchungen über das logische Schliessen," *Mathematische Zeitschrift* 39 (1934): 176–210, 405–431.
2. For more on logical equivalence, see section 7.5.
3. The most famous intuitionist is the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966). See Anthony Flew, *A Dictionary of Philosophy* (New York: St. Martin's Press, 1979), p. 178.
4. The form of the argument and the observation that it cannot be proved directly from the rules of inference adopted thus far are borrowed from Howard Kahane, *Logic and Philosophy: A Modern Introduction*, 6th ed. (Belmont, CA: Wadsworth, 1990), p. 88.
5. This observation is borrowed from Kahane, *Logic and Philosophy*, p. 88. A popular text that uses absorption instead of CP is Irving M. Copi and Carl Cohen, *Introduction to Logic*, 8th ed. (New York: Macmillan, 1990), chap. 9.