Chapter 8

Statement Logic: Proofs

In the previous chapter, we used truth-tables to evaluate arguments in statement logic (the part of logic in which atomic statements are the basic units). We saw, however, that truth tables are cumbersome when applied to arguments involving numerous statement letters. In this chapter, we will develop a system of natural deduction that has certain advantages over the truth table method. In a system of *natural deduction*, one uses a set of inference rules to prove that the conclusion of an argument follows from its premises. And for the purpose of proving arguments valid, a system of natural deduction has at least two advantages over the truth table method. First, it is less cumbersome. Second, such systems more clearly mirror our intuitive patterns of reasoning (the ways we ordinarily argue) than do truth tables. The German logician and mathematician Gerhard Gentzen (1909–1945) was the first to develop a system of natural deduction.[1]

Our system of natural deduction will be introduced in stages. Section 8.1 gives us 8 initial rules of inference that permit us to construct a limited variety of proofs. Sections 8.2 and 8.3 each introduce 5 more rules, bringing the total to 18 rules. In section 8.4, we add a special rule called "conditional proof." With the addition of conditional proof, our system of natural deduction can prove as valid any argument that is valid according to the truth table method. And since each of our rules of inference is itself valid, *any argument that can be proved valid in our system of natural deduction is indeed valid*. In section 8.5, we add one more rule, called "*reductio ad absurdum*," that makes many proofs either shorter or more intuitive. Finally, in section 8.6, we discuss proving theorems.

8.1 Implicational Rules of Inference

Let us use the word "proof" in a technical sense to refer to a series of steps that leads from the premises of a symbolic argument to its conclusion. The fundamental idea is to show that the premises lead, by way of valid rules of inference, to the conclusion. The underlying principle is this: *Whatever follows from a set of statements by means of valid inferences is true if all the statements in the set are true.*

Our first set of inference rules is mostly familiar. The first five were introduced as argument forms in Chapter 1. Once again, we use italicized, lowercase letters as variables that stand for any given statement: ***p***, ***q***, ***r***, and ***s***.

Rule 1: *Modus ponens* (MP):

$$\begin{array}{l} p \rightarrow q \\ p \\ \therefore q \end{array}$$

Rule 2: *Modus tollens* (MT):

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \therefore \sim p \end{array}$$

Rule 3: Hypothetical syllogism (HS):

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \therefore p \rightarrow r \end{array}$$

Rule 4: Disjunctive syllogism (DS), in two forms:

$$\begin{array}{lcl} p \vee q & \qquad & p \vee q \\ \sim p & & \sim q \\ \therefore q & & \therefore p \end{array}$$

Rule 5: Constructive dilemma (CD):

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow s \\ \therefore r \vee s \end{array}$$

Note that each rule is given an abbreviation designed to cut down on the amount of writing involved in constructing proofs. To these familiar forms we add three additional patterns of inference, two involving conjunctions and one involving disjunctions.

Rule 6: Simplification (Simp), in two forms:

$$\begin{array}{lcl} p \bullet q & \qquad & p \bullet q \\ \therefore p & & \therefore q \end{array}$$

Simplification says, in effect, that if you have a conjunction, then you may infer either conjunct. Here is an English example:

1. Both Pierre Curie and Marie Curie were physicists. Therefore, Marie Curie was a physicist.

This type of inference may seem so obvious as to be trivial, but it is nonetheless valid. And one aspect of the power of logic is its capacity to break complex reasoning down into easy steps.

The next rule tells us that if we have two statements as steps in an argument, we may conjoin them.

Rule 7: Conjunction (Conj):

$$\begin{array}{l} p \\ q \\ \therefore p \cdot q \end{array}$$

Again, this rule is obviously valid. Here is an example:

2. Thomas Aquinas died in 1274. William Ockham died in 1349. Consequently, Aquinas died in 1274, *and* Ockham died in 1349.

The rule of addition is perhaps a bit less obvious than the rules we have considered so far.

Rule 8: Addition (Add) in two forms:

$$\begin{array}{l} p \\ \therefore p \vee q \end{array} \qquad \begin{array}{l} p \\ \therefore q \vee p \end{array}$$

This schema tells us that from any given statement p, one may infer a disjunction that has p as one of its disjuncts—and the other disjunct may be anything you please. For instance:

3. Thomas Paine wrote *Common Sense*. Hence, either Thomas Paine wrote *Common Sense* or Patrick Henry wrote *Common Sense* (or both did).

This type of inference may seem odd, but it is valid. Recall that only one disjunct must be true for an inclusive disjunction to be true. Thus, "Either 1 + 1 = 2 or 2 + 2 = 22 (or both)" is true, even though "2 + 2 = 22" is false. And hence, every instance of addition is valid since it is impossible for the conclusion of an argument of this form to be false *given that its premise is true*.

Is the following argument an example of addition?

4. Adam stole the money. It follows that either Adam stole the money or Betty stole the money, but not both. (A: Adam stole the money; B: Betty stole the money)

No. Argument (4) has the following invalid form:

5. $A \ \therefore (A \vee B) \cdot \sim(A \cdot B)$

And it is easy to construct a counterexample to this pattern of reasoning. For instance:

6. Four is an even number. So, either 4 is even or 6 is even, *but not both*.

Because the premise is obviously true but the conclusion is obviously false, the counterexample proves that form (5) is invalid. So, it is important not to confuse form (5) with the rule of addition.

The italicized, lowercase letters in the previous argument schemas play a special role. They can be replaced by any symbolic sentence *as long as the replacement is uniform throughout the argument*. For example, both of the following count as instances of *modus ponens*:

$$
\begin{array}{llcll}
 & \sim F \rightarrow G & \quad & & L \rightarrow (M \rightarrow N) \\
 & \sim F & & & L \\
\therefore & G & & \therefore & M \rightarrow N
\end{array}
$$

In the inference on the left, $\sim F$ is substituted for the letter p, while G is substituted for the letter q in the original schema: $p \rightarrow q, p \therefore q$. Note that we have replaced p with $\sim F$ throughout the argument schema; substitutions must be uniform in this sense. In the example on the right, L is substituted for p, while $(M \rightarrow N)$ is substituted for q in the original schema. In both cases, the pattern of reasoning is *modus ponens* because one premise is a conditional, the other is the antecedent of the conditional, and the conclusion is the consequent of the conditional.

In substituting symbolic formulas for lowercase letters, precision is required. Consider the following argument. Is it an instance of *modus tollens*?

$$
\begin{array}{ll}
 & C \rightarrow \sim D \\
 & D \\
\therefore & \sim C
\end{array}
$$

No, it is not. The schema for *modus tollens* is $p \rightarrow q, \sim q \therefore \sim p$. If we replace the letter q with $\sim D$ in the first premise, we must replace q with $\sim D$ in the second premise as well, in which case we obtain the following argument:

$$
\begin{array}{ll}
 & C \rightarrow \sim D \\
 & \sim\sim D \\
\therefore & \sim C
\end{array}
$$

This *is* an instance of *modus tollens*. To apply *modus tollens*, we need a conditional *and* the negation of its consequent. If the consequent of the conditional is itself a negation, such as $\sim D$, the other premise will be a double-negation, such as $\sim\sim D$ above.

To ensure an understanding of our new inference rules, let us consider a series of examples. Which rules of inference, if any, are exemplified by the following arguments?

$$
\begin{array}{llcll}
 & \sim P \rightarrow (Q \bullet R) & \quad & & X \vee (Y \leftrightarrow Z) \\
 & (Q \bullet R) \rightarrow S & & & \sim(Y \leftrightarrow Z) \\
\therefore & \sim P \rightarrow S & & \therefore & X
\end{array}
$$

The argument on the left is an example of hypothetical syllogism. Note that $\sim P$ replaces p, $(Q \cdot R)$ replaces q, and S replaces r in the original schema: $p \rightarrow q$, $q \rightarrow r \therefore p \rightarrow r$. The argument on the right is an example of disjunctive syllogism. Here, X replaces p and $(Y \leftrightarrow Z)$ replaces q in the second form of disjunctive syllogism: $p \vee q$, $\sim q \therefore p$.

Which rules of inference, if any, are exemplified by the following arguments?

$$\begin{array}{ll} & \sim M \vee \sim N \\ & \sim M \rightarrow \sim O \\ & \sim N \rightarrow \sim P \\ \therefore & \sim O \vee \sim P \end{array} \qquad \begin{array}{ll} & \sim(B \cdot \sim C) \\ \therefore & \sim(B \cdot \sim C) \vee \sim D \end{array}$$

The argument on the left is an example of constructive dilemma. Here, $\sim M$ replaces p, $\sim N$ replaces q, $\sim O$ replaces r, and $\sim P$ replaces s in the schema for constructive dilemma: $p \vee q$, $p \rightarrow r$, $q \rightarrow s \therefore r \vee s$. The argument on the right is an example of addition. Note that $\sim(B \cdot \sim C)$ replaces p, while $\sim D$ replaces q in the first form of addition: $p \therefore p \vee q$.

Which rules of inference, if any, are exemplified by the following arguments?

$$\begin{array}{ll} & (C \rightarrow D) \cdot (E \vee F) \\ \therefore & E \vee F \end{array} \qquad \begin{array}{ll} & A \vee \sim B \\ & B \\ \therefore & A \end{array}$$

The argument on the left is an example of simplification. Here, $(C \rightarrow D)$ replaces p, while $(E \vee F)$ replaces q in the second form of simplification: $p \cdot q \therefore q$. The argument on the right, however, does *not* exemplify any of our inference rules. But if we changed the second premise to $\sim\sim B$, then we would have an instance of the second form of disjunctive syllogism: $p \vee q$, $\sim q \therefore p$. (Substitute A for p and $\sim B$ for q.)

Let us now use our new inference rules to construct some proofs. We begin with an English argument:

7. If some employees deserve 5 times the wages of others, then some employees are 5 times more valuable than others. It is not true that some employees are 5 times more valuable than others. So, it is not true that some employees deserve 5 times the wages of others. (D: Some employees deserve 5 times the wages of others; V: Some employees are 5 times more valuable than others)

Using the scheme of abbreviation provided, the argument should be symbolized as follows:

1. $D \rightarrow V$
2. $\sim V \qquad \therefore \sim D$

The first lines of our proof contain the premises of the argument. To the right of the last premise we write the conclusion, marked by the three-dot symbol. This serves as a reminder of what we are trying to derive from the premises. (Thus, the expression $\therefore \sim D$ is not really a part of the proof but merely a reminder of what we need to prove.) What we want to do is to arrive at the conclusion, $\sim D$, by means of our inference rules. We have a conditional premise, and we also have the negation of its consequent. That is, we have here the makings of a *modus tollens*–type argument. To see this, substitute D for p and substitute V for q in the original diagram for *modus tollens*: $p \rightarrow q, \sim q \therefore \sim p$. Proper proof procedure requires that *we list the lines to which we are applying the rule of inference, as well as the abbreviation of the inference rule*. Accordingly, our completed proof looks like this:

1.	$D \rightarrow V$	
2.	$\sim V$	$\therefore \sim D$
3.	$\sim D$	1, 2, MT

Line (3) tells us that $\sim D$ follows from lines (1) and (2) by *modus tollens*. We have shown that the premises of the argument lead to the conclusion by way of a valid rule of inference. Notice that the only lines in the proof without annotation (without an explicit indication of how we arrived at them) are the premises. *Let us adopt the convention that any step in an argument without annotation will be understood to be a premise*.

Consider a slightly more complicated example:

8. If the workplace is a meritocracy, then the most qualified person always gets the job. But the most qualified person does not always get the job if networking plays a role in who gets most jobs. Furthermore, networking does play a role in who gets most jobs. Therefore, the workplace is not a meritocracy. (W: The workplace is a meritocracy; M: The most qualified person always gets the job; N: Networking plays a role in who gets most jobs)

Using the scheme of abbreviation provided, the argument should be symbolized as follows:

1.	$W \rightarrow M$	
2.	$N \rightarrow \sim M$	
3.	N	$\therefore \sim W$

As before, the first lines of our proof contain the premises of the argument, with the conclusion written off to the right of the last premise. The completed proof runs as follows:

1.	$W \rightarrow M$	
2.	$N \rightarrow \sim M$	
3.	N	$\therefore \sim W$
4.	$\sim M$	2, 3, MP
5.	$\sim W$	1, 4, MT

Lines (2) and (3) imply ~M by the rule *modus ponens*. To see this, replace p with N and q with ~M in the schema for *modus ponens*: $p \rightarrow q, p \therefore q$. Lines (1) and (4) imply ~*W* by the rule *modus tollens*: $p \rightarrow q, \sim q \therefore \sim p$ (replacing p with *W* and q with M).

Let us now consider a proof that employs our inference rules involving conjunctions:

9. Women earn only 75¢ for every dollar earned by men. If women earn only 75¢ for every dollar earned by men, and 90% of children who live with one parent live with their mothers, then men are better off than women, and women are victims of injustice. Ninety percent of children who live with one parent live with their mothers. Feminists are right if women are victims of injustice. So, feminists are right. (W: Women earn only 75¢ for every dollar earned by men; C: 90% of children who live with one parent live with their mothers; M: Men are better off than women; V: Women are victims of injustice; F: Feminists are right)

Using the scheme of abbreviation provided, the translation into symbols looks like this:

1. W
2. (W • C) → (M • V)
3. C
4. V → F   ∴ F

The proof may be completed thus:

5. W • C	1, 3, Conj
6. M • V	2, 5, MP
7. V	6, Simp
8. F	4, 7, MP

Note that line (5) comes from lines (1) and (3) by substituting *W* for p and C for q in the schema for conjunction: $p, q \therefore p \bullet q$. And line (7) comes from line (6) by substituting M for p and *V* for q in the second form of simplification: $p \bullet q \therefore q$.

One last example will demonstrate some of the inference rules involving disjunctions.

10. If Pierre is an assassin, then either he should be put to death, or he should be given a life sentence. He should be put to death only if murderers deserve death. He should be given a life sentence only if murderers forfeit their right to liberty. Pierre is an assassin, but murderers do not deserve death. Therefore, murderers forfeit their right to liberty. (A: Pierre is an assassin; D: Pierre should be put to death; L: Pierre should be given a life sentence; M: Murderers deserve death; F: Murderers forfeit their right to liberty)

Using the scheme of abbreviation provided, the argument may be symbolized like this:

1. $A \rightarrow (D \vee L)$
2. $D \rightarrow M$
3. $L \rightarrow F$
4. $A \bullet {\sim}M \quad \therefore\ F$

The proof may be completed as follows:

5. A	4, Simp
6. $D \vee L$	1, 5, MP
7. $M \vee F$	6, 2, 3, CD
8. ${\sim}M$	4, Simp
9. F	7, 8, DS

Note that line (7) derives from the steps indicated by substituting *D* for ***p***, *L* for ***q***, M for ***r***, and *F* for ***s*** in the original schema for constructive dilemma: $p \vee q$, $p \rightarrow r$, $q \rightarrow s \therefore r \vee s$. And line (9) derives from lines (7) and (8) by substituting M for ***p*** and F for ***q*** in the first form of disjunctive syllogism: $p \vee q$, ${\sim}p \therefore q$.

Our first eight inference rules are called **implicational rules** to set them apart from equivalence rules, which will be introduced in the next section. When using an equivalence rule, one always moves from a single statement (such as ${\sim}H \bullet J$) to another statement that is logically equivalent to the first (such as $J \bullet {\sim}H$). (Recall that logically equivalent statements validly imply each other. They agree in truth value regardless of the truth values assigned to their atomic components.) But implicational rules lack this feature. For example, we may move from $F \bullet G$ to F by simplification, but obviously F is not logically equivalent to $F \bullet G$. Because of this difference between implicational and equivalence rules, implicational rules must be applied to entire lines in a proof and not merely to parts of lines. To illustrate, consider the following inference:

1. $F \rightarrow (G \rightarrow H)$
2. ${\sim}H \quad \therefore\ F \rightarrow {\sim}G$
3. $F \rightarrow {\sim}G \quad$???

Does line (3) follow from lines (1) and (2) by *modus tollens*? No. *Modus tollens* must be applied to entire lines in a proof. So, in order to have an instance of *modus tollens*, we need two things:

- A conditional statement that is an entire line in a proof
- Another line that is *the negation of the consequent* of that conditional

In the preceding example, the first is satisfied but the second is not. Line (1) *is* a conditional that represents an entire line in the proof. But the negation of its consequent is $\sim(G \rightarrow H)$, not $\sim H$. So, line (3) does not follow from lines (1) and (2) by *modus tollens*.

Notice that we construct our proofs by means of *one* application of *one* rule of inference per line, as indicated in the annotation listed to the right. We do this to ensure that each step in every proof is explicit and clearly justified by a rule in our system of logic. Which of the following proofs is properly constructed?

1. $A \rightarrow \sim B$		1. $A \rightarrow \sim B$	
2. $A \bullet C$	$\therefore \sim B$	2. $A \bullet C$	$\therefore \sim B$
3. $\sim B$	1, 2, MP	3. A	2, Simp
		4. $\sim B$	1, 3, MP

The proof on the right is correct; the proof on the left skips a required application of simplification and misapplies *modus ponens*. To apply *modus ponens*, we need (a) a conditional that is an entire line in the proof and (b) another line that is *precisely* the antecedent of that conditional. Clause (a) is satisfied in line (1) of the above proofs, but clause (b) is not satisfied in the proof on the left because A never appears by itself on any line.

The following hints or rules of thumb may help you as you construct proofs:

> **Rule of Thumb 1**: It usually helps to work backward. So, start by looking at the conclusion, and then try to find the conclusion (or elements thereof) in the premises.

For example:

1. $A \rightarrow [B \rightarrow (C \vee D)]$
2. $B \bullet A$
3. $\sim D$ $\qquad\qquad \therefore C$

The conclusion here is C. Does it appear anywhere in the premises? Yes, it is embedded in the consequent of premise (1). And if we could obtain $C \vee D$ from premise (1), we could combine it with $\sim D$—that is, premise (3)—to get C, by disjunctive syllogism. But how can we obtain $C \vee D$? Consider a second rule of thumb:

> **Rule of Thumb 2**: Apply the inference rules to break down the premises.

We could get A from line (2) by simplification and use it together with line (1) to obtain $B \rightarrow (C \vee D)$, by *modus ponens*. Then we could get B from line (2) (by simplification) and apply *modus ponens* again, to obtain $C \vee D$. The whole proof would then look like this:

1. $A \rightarrow [B \rightarrow (C \vee D)]$
2. $B \bullet A$
3. $\sim D$ $\quad\therefore C$
4. A $\quad$ 2, Simp
5. $B \rightarrow (C \vee D)$ $\quad$ 1, 4, MP
6. B $\quad$ 2, Simp
7. $C \vee D$ $\quad$ 5, 6, MP
8. C $\quad$ 3, 7, DS

Let's consider another example:

1. $E \vee F$
2. $E \rightarrow G$
3. $F \rightarrow H$
4. $(G \vee H) \rightarrow J$ $\quad\therefore J \vee K$

Using rule of thumb 1, we start by examining the conclusion. We look to see if the conclusion (or parts thereof) appear in the premises, noting that *J* is the consequent of premise (4). Now, is there any way to break premise (4) down, as rule of thumb 2 suggests? Yes, we can use the rule of constructive dilemma to obtain $G \vee H$ from premises (1), (2), and (3), and then use *modus ponens* to get *J*. But where do we go from there? In particular, how can we obtain *K* when it appears nowhere in the premises? At this point, it will be helpful to bear in mind an additional rule of thumb:

> **Rule of Thumb 3:** If the conclusion contains a statement letter that does not appear in the premises, use the rule of addition.

The whole proof looks like this:

1. $E \vee F$
2. $E \rightarrow G$
3. $F \rightarrow H$
4. $(G \vee H) \rightarrow J$ $\quad\therefore J \vee K$
5. $G \vee H$ $\quad$ 1, 2, 3, CD
6. J $\quad$ 4, 5, MP
7. $J \vee K$ $\quad$ 6, Add

These rules of thumb are helpful in many cases, but a certain amount of ingenuity is required when doing proofs. Furthermore, to gain facility in constructing proofs, one must practice. Hence, the following exercises are provided.

Summary of Implicational Rules

1. *Modus ponens* (MP): $\begin{array}{rl} & p \rightarrow q \\ & p \\ \therefore & q \end{array}$

2. *Modus tollens* (MT): $\begin{array}{rl} & p \rightarrow q \\ & \sim q \\ \therefore & \sim p \end{array}$

3. Hypothetical syllogism (HS): $\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \therefore & p \rightarrow r \end{array}$

4. Disjunctive syllogism (DS), in two forms: $\begin{array}{rl} & p \vee q \\ & \sim p \\ \therefore & q \end{array}$ $\quad \begin{array}{rl} & p \vee q \\ & \sim q \\ \therefore & p \end{array}$

5. Constructive dilemma (CD): $\begin{array}{rl} & p \vee q \\ & p \rightarrow r \\ & q \rightarrow s \\ \therefore & r \vee s \end{array}$

6. Simplification (Simp), in two forms: $\begin{array}{rl} & p \bullet q \\ \therefore & p \end{array}$ $\quad \begin{array}{rl} & p \bullet q \\ \therefore & q \end{array}$

7. Conjunction (Conj): $\begin{array}{rl} & p \\ & q \\ \therefore & p \bullet q \end{array}$

8. Addition (Add), in two forms: $\begin{array}{rl} & p \\ \therefore & p \vee q \end{array}$ $\quad \begin{array}{rl} & p \\ \therefore & q \vee p \end{array}$

Exercise 8.1

Part A: Annotating For each of the following proofs, indicate from which steps each inference is drawn and by which rule the inference is made. (See the Answer Key for an illustration.)

* **1.** 1. F → G
2. G → H ∴ F → H
3. F → H

2. 1. ~S → ~P
2. ~S ∴ K ∨ ~P
3. ~P
4. K ∨ ~P

3. 1. E → (T → S)
2. ~(T → S)
3. ~R ∨ E ∴ ~R
4. ~E
5. ~R

* **4.** 1. $H \vee \sim C$
2. $H \rightarrow \sim B$
3. $\sim C \rightarrow D$
4. $(\sim B \vee D) \rightarrow (K \cdot J)$ $\therefore J$
5. $\sim B \vee D$
6. $K \cdot J$
7. J

5. 1. D
2. $\sim H$
3. $(D \cdot \sim H) \rightarrow (E \vee H)$ $\therefore E$
4. $D \cdot \sim H$
5. $E \vee H$
6. E

6. 1. $\sim A \rightarrow \sim B$
2. $\sim\sim B \cdot C$ $\therefore \sim\sim A \cdot C$
3. $\sim\sim B$
4. $\sim\sim A$
5. C
6. $\sim\sim A \cdot C$

* **7.** 1. $\sim(P \cdot Q) \vee R$
2. $(\sim E \cdot \sim R) \rightarrow (A \cdot B)$
3. $E \rightarrow (P \cdot Q)$
4. $\sim R$ $\therefore B \vee (F \cdot G)$
5. $\sim(P \cdot Q)$
6. $\sim E$
7. $\sim E \cdot \sim R$
8. $A \cdot B$
9. B
10. $B \vee (F \cdot G)$

8. 1. $F \vee S$
2. G
3. $[G \cdot (F \vee S)] \rightarrow \sim T$
4. $\sim B \rightarrow T$ $\therefore \sim\sim B$
5. $G \cdot (F \vee S)$
6. $\sim T$
7. $\sim\sim B$

9. 1. $F \rightarrow B$
2. $\sim D$
3. $(\sim D \cdot G) \rightarrow (B \rightarrow S)$
4. G
5. $\sim S$ $\therefore G \cdot \sim F$
6. $\sim D \cdot G$
7. $B \rightarrow S$
8. $F \rightarrow S$
9. $\sim F$
10. $G \cdot \sim F$

* **10.** 1. $W \rightarrow (X \vee \sim Y)$
2. $\sim\sim Y \cdot W$ $\therefore X \vee \sim Z$
3. W
4. $X \vee \sim Y$
5. $\sim\sim Y$
6. X
7. $X \vee \sim Z$

Part B: Correct or Incorrect? Some of the following inferences are correct applications of the eight rules introduced in this section, and some are not. If an inference is a correct application of a rule, name the rule. If an inference is not a correct application of a rule, explain why it is not. (The question is whether the conclusion in each case can be reached *in a single step* from the premise(s) by an application of one of the rules.)

* **1.** $K \rightarrow L$
$\sim L$
$\therefore \sim K$

2. $G \rightarrow F$
$E \rightarrow G$
$\therefore E \rightarrow F$

3. $M \rightarrow N$
$\sim M$
$\therefore \sim N$

* **4.** $\sim B \vee \sim Y$
$\sim B \rightarrow \sim X$
$\sim Y \rightarrow Z$
$\therefore \sim X \vee Z$

5. $(E \cdot F) \vee G$
$\sim(E \cdot F)$
$\therefore G$

6. $(N \cdot P) \to (O \vee S)$
$O \vee S$
$\therefore N \cdot P$

* 7. $\sim E \vee \sim F$
$\sim\sim F$
$\therefore \sim E$

8. $A \to \sim\sim B$
$\sim B$
$\therefore \sim A$

9. $(P \to Q) \vee R$
P
$\therefore Q \vee R$

* 10. $(R \cdot S) \to T$
$\therefore S \to T$

11. $(K \vee L) \to M$
$\sim M$
$\therefore \sim(K \vee L)$

12. $(H \vee \sim S) \to \sim W$
$H \vee \sim S$
$\therefore \sim W$

* 13. $T \to \sim U$
U
$\therefore \sim T$

14. $P \cdot Q$
$\therefore (R \to S) \vee (P \cdot Q)$

15. $R \to \sim S$
$\sim\sim S$
$\therefore \sim R$

* 16. $\sim C \vee \sim D$
$X \to C$
$Y \to D$
$\therefore \sim X \vee \sim Y$

17. $\sim G \vee \sim P$
G
$\therefore \sim P$

18. $A \to (B \to C)$
B
$\therefore A \to C$

* 19. $(\sim B \vee D) \to E$
$\sim\sim B$
$\therefore D \to E$

20. $\sim T \to \sim N$
$\sim\sim N$
$\therefore \sim\sim T$

Part C: Proofs Construct proofs to show the following symbolic arguments valid. Commas mark the breaks between premises. (See the Answer Key for an illustration.)

* 1. $H \to \sim B,\ D \to B,\ H \ \therefore \sim D$

2. $F \to (G \to H),\ \sim F \to J,\ \sim(G \to H) \ \therefore J$

3. $(F \vee E) \to \sim D,\ S \vee D,\ E \ \therefore S$

* 4. $\sim A \to F,\ A \to D,\ \sim D,\ F \to S \ \therefore S \vee X$

5. $(A \cdot E) \to F,\ E,\ F \to (D \cdot \sim C),\ A \ \therefore \sim C$

6. $\sim F \vee \sim G,\ \sim F \to Z,\ \sim G \to \sim R,\ (Z \vee \sim R) \to (U \to P),\ \sim P \ \therefore \sim U$

* 7. $\sim(S \vee R),\ B \to (S \vee R),\ B \vee P,\ \sim Q \vee B \ \therefore P \cdot \sim Q$

8. $C \to (T \to L),\ \sim L,\ \sim E \to C,\ L \vee \sim E \ \therefore \sim T$

9. $\sim\sim A,\ B \to \sim A \ \therefore \sim B$

*10. $(B \cdot A) \rightarrow C,\ \sim D \rightarrow (B \cdot A),\ \sim C\ \therefore\ \sim\sim D$

11. $(\sim B \cdot \sim C) \rightarrow (D \rightarrow C),\ \sim B,\ C \rightarrow B\ \therefore\ \sim D$

12. $(D \cdot H) \rightarrow R,\ S \rightarrow (D \cdot H)\ \therefore\ S \rightarrow R$

*13. $(T \rightarrow C) \rightarrow \sim F,\ S \rightarrow C,\ T \rightarrow S,\ F \vee \sim P\ \therefore\ \sim P$

14. $(A \vee \sim B) \rightarrow (F \vee (R \cdot G)),\ A,\ F \rightarrow L,\ (R \cdot G) \rightarrow T,\ (L \vee T) \rightarrow S\ \therefore\ S$

15. $P \vee Q,\ (Q \cdot \sim R) \rightarrow S,\ R \rightarrow P,\ \sim P\ \therefore\ S$

*16. $(E \vee F) \rightarrow \sim G,\ \sim H,\ H \vee K,\ (K \vee L) \rightarrow E\ \therefore\ \sim G$

17. $(M \vee N) \rightarrow \sim S,\ T \rightarrow (M \vee N),\ \sim S \rightarrow \sim(M \vee N)\ \therefore\ T \rightarrow \sim(M \vee N)$

18. $(E \vee \sim B) \rightarrow (\sim S \vee T),\ E,\ \sim S \rightarrow L,\ T \rightarrow \sim C,\ (L \vee \sim C) \rightarrow A\ \therefore\ A$

*19. $\sim\sim B,\ \sim C \rightarrow \sim B,\ (\sim\sim C \vee T) \rightarrow P\ \therefore\ P$

20. $B \vee \sim C,\ B \rightarrow E,\ \sim\sim C\ \therefore\ \sim B \vee E$

Part D: More Proofs Construct proofs to show that the following arguments are valid. Commas mark the breaks between premises.

* 1. $P \rightarrow Q,\ R \rightarrow \sim S,\ P \vee R,\ (Q \vee \sim S) \rightarrow (\sim T \vee \sim W),\ \sim\sim T\ \therefore\ \sim W$

2. $(A \vee G) \rightarrow K,\ K \rightarrow (B \rightarrow F),\ A \cdot B\ \therefore\ F$

3. $\sim M,\ (\sim M \cdot \sim N) \rightarrow (Q \rightarrow P),\ \sim N,\ P \rightarrow R\ \therefore\ Q \rightarrow R$

* 4. $\sim(R \vee S),\ \sim(T \cdot V) \rightarrow (R \vee S),\ \sim\sim(T \cdot V) \rightarrow W\ \therefore\ W \vee \sim R$

5. $\sim W \cdot \sim\sim Z,\ (\sim W \cdot X) \rightarrow Y,\ \sim Z \vee X\ \therefore\ Y$

6. $F \rightarrow A,\ \sim J \cdot \sim K,\ H \rightarrow (G \rightarrow F),\ \sim K \rightarrow (\sim J \rightarrow H)\ \therefore\ G \rightarrow A$

* 7. $\sim F \rightarrow J,\ \sim F \vee \sim G,\ \sim G \rightarrow \sim H,\ (J \vee \sim H) \rightarrow \sim K,\ \sim L \rightarrow K\ \therefore\ \sim\sim L$

8. $Y \rightarrow W,\ (Z \rightarrow W) \rightarrow (V \cdot \sim T),\ Z \rightarrow Y,\ Q \rightarrow T\ \therefore\ \sim Q$

9. $(\sim N \cdot M) \rightarrow T,\ \sim O \rightarrow M,\ \sim O \cdot \sim N\ \therefore\ T \vee S$

*10. $\sim A \cdot \sim C,\ \sim C \rightarrow D,\ (D \cdot \sim A) \rightarrow (E \rightarrow \sim H),\ E \cdot (\sim F \rightarrow H)\ \therefore\ \sim\sim F$

11. $R \rightarrow D,\ B \rightarrow R,\ (B \rightarrow D) \rightarrow (E \vee F),\ \sim E\ \therefore\ F$

12. $\sim F \rightarrow \sim G,\ P \rightarrow \sim Q,\ \sim F \vee P,\ (\sim G \vee \sim Q) \rightarrow (L \cdot M)\ \therefore\ L$

*13. $(Z \cdot A) \vee \sim Y,\ (Z \cdot A) \rightarrow U,\ W \vee \sim U,\ \sim W\ \therefore\ \sim Y$

14. $(D \cdot E) \vee F,\ F \rightarrow C,\ (D \cdot E) \rightarrow \sim B,\ (\sim B \vee C) \rightarrow (A \rightarrow P),\ \sim P\ \therefore\ \sim A$

15. $O \rightarrow N,\ \sim M,\ S \rightarrow R,\ P \rightarrow O,\ R \rightarrow P,\ (S \rightarrow N) \rightarrow (M \vee L)\ \therefore\ \sim P \vee L$

*16. $(\sim M \vee L) \rightarrow (\sim A \rightarrow B),\ \sim S \rightarrow T,\ R \rightarrow \sim S,\ \sim M \cdot J,\ R \vee \sim A\ \therefore\ T \vee B$

17. $F \rightarrow \sim G,\ \sim\sim G \cdot O,\ (\sim F \cdot \sim\sim G) \rightarrow [(\sim H \rightarrow E) \cdot (C \rightarrow F)],\ C \vee \sim H$
$\therefore\ E$

18. $\sim N \cdot \sim M,\ \sim P \rightarrow N,\ \sim N \rightarrow Z,\ (Z \cdot \sim\sim P) \rightarrow K\ \therefore\ K \cdot \sim M$

*19. $A \vee D,\ \sim D,\ (C \vee A) \rightarrow \sim E\ \therefore\ \sim E$

20. $(C \rightarrow Q) \cdot (\sim L \rightarrow \sim R)$, $(S \rightarrow C) \cdot (\sim N \rightarrow \sim L)$, $\sim Q \cdot J$, $\sim Q \rightarrow (S \vee \sim N)$ $\therefore \sim R$

21. $\sim(Z \vee Y) \rightarrow \sim W$, $\sim U \rightarrow \sim(Z \vee Y)$, $(\sim U \rightarrow \sim W) \rightarrow (T \rightarrow S)$, $S \rightarrow (R \vee P)$, $[T \rightarrow (R \vee P)] \rightarrow [(\sim R \vee K) \cdot \sim K]$ $\therefore \sim R$

* 22. $\sim A$, $[\sim A \vee (B \cdot C)] \rightarrow (D \rightarrow \sim E)$, $\sim E \rightarrow \sim F$, $(D \rightarrow \sim F) \rightarrow G$ $\therefore (G \cdot \sim A) \vee \sim H$

23. $(S \vee U) \cdot \sim U$, $S \rightarrow [T \cdot (F \vee G)]$, $[T \vee (J \cdot P)] \rightarrow (\sim B \cdot E)$ $\therefore S \cdot \sim B$

24. $\sim X \rightarrow (\sim Y \rightarrow \sim Z)$, $X \vee (W \rightarrow U)$, $\sim Y \vee W$, $\sim X \cdot T$, $(\sim Z \vee U) \rightarrow \sim S$ $\therefore (R \vee \sim S) \cdot T$

* 25. $(D \vee C) \rightarrow (F \vee H)$, $(H \cdot G) \rightarrow (F \vee E)$, $(D \vee B) \rightarrow (\sim F \rightarrow G)$, $(F \vee D) \cdot (\sim F \cdot A)$ $\therefore E$

Part E: English Arguments Symbolize the following arguments using the schemes of abbreviation provided. Then construct proofs to show that the arguments are valid.

* 1. No one can know anything, for every piece of reasoning must start somewhere. And if every piece of reasoning must start somewhere, then every piece of reasoning begins with an unsupported premise. Now, if every piece of reasoning begins with an unsupported premise, then all human thinking is based on mere assumption. And if all human thinking is based on mere assumption, no one can know anything. (S: Every piece of reasoning must start somewhere; U: Every piece of reasoning begins with an unsupported premise; A: All human thinking is based on mere assumption; K: No one can know anything)

2. Theists say that God created the world. They say that the world must have a cause. But why? The world must have a cause only if everything must have a cause. But if everything must have a cause, then God has a cause. However, God isn't God if God has a cause. And if God isn't God, God doesn't exist. So, if the world must have a cause, there is no God. (W: The world must have a cause; E: Everything must have a cause; H: God has a cause; G: God is God; X: God exists)

3. Either we should stop going places, or we should develop hydrogen-powered cars, or we should go on driving gasoline-powered cars. We should go on driving gasoline-powered cars only if we should destroy the ozone layer. We should not stop going places and we should not destroy the ozone layer. Therefore, we should develop hydrogen-powered cars. (P: We should stop going places; H: We should develop hydrogen-powered cars; G: We should go on driving gasoline-powered cars; O: We should destroy the ozone layer)

* 4. Dinosaurs are extinct. And given that dinosaurs are extinct, they suffered some catastrophe if they died suddenly. The dinosaurs died suddenly, assuming that they froze due to a sudden drop in temperature or were attacked by a lethal virus. The dinosaurs froze due to a sudden drop in temperature

provided that the sun's rays were blocked. The earth's atmosphere was filled with dust due to the impact of a comet, and the sun's rays were blocked. Therefore, the dinosaurs suffered some catastrophe. (E: Dinosaurs are extinct; C: The dinosaurs suffered some catastrophe; D: The dinosaurs died suddenly; F: The dinosaurs froze due to a sudden drop in temperature; V: The dinosaurs were attacked by a lethal virus; S: The sun's rays were blocked; A: The earth's atmosphere was filled with dust due to the impact of a comet)

5. In spite of the fact that advocates of suicide and euthanasia often claim that every right—including the right to life—can be waived, I think it's absurd to suggest that every right can be waived. (To waive a right is to agree, for good moral reasons, not to exercise it.) For if every right can be waived, then if I announce that I am waiving my right to liberty, you are morally permitted to enslave me. But obviously, it is not true that if I announce that I am waiving my right to liberty, then you are morally permitted to enslave me. (E: Every right can be waived; A: I announce that I am waiving my right to liberty; P: You are morally permitted to enslave me)

6. If morality is not subjective, then either morality is relative to cultures, or God is the source of all moral values. If morality is subjective, then if I approve of racism, racism is right. Plainly, it's false that if I approve of racism, racism is right. Furthermore, if morality is relative to cultures, then the cannibalism in New Guinea is right, and the caste system in India is right. The statement "The cannibalism in New Guinea is right, and the caste system in India is right" is false. God exists if God is the source of all moral values. Accordingly, God exists. (S: Morality is subjective; M: Morality is relative to cultures; V: God is the source of all moral values; A: I approve of racism; R: Racism is right; C: The cannibalism in New Guinea is right; I: The caste system in India is right; G: God exists)

* 7. Al has precognition. And assuming that Al has precognition, Al experiences events prior to their occurrence. But if Al experiences events prior to their occurrence, then either events exist prior to their occurrence, or Al predicts the future on the basis of what he knows about the past and present. It is simply nonsense to say that events exist prior to their occurrence. We may infer that Al predicts the future on the basis of what he knows about the past and present. (P: Al has precognition; A: Al experiences events prior to their occurrence; E: Events exist prior to their occurrence; F: Al predicts the future on the basis of what he knows about the past and present)

8. God's existence is either necessary or impossible, if it is not contingent. God's existence is a matter of metaphysical luck if it is contingent. God's existence is emphatically not a matter of metaphysical luck. God's existence is not impossible if the concept of an omnipotent and perfectly good being is coherent. The concept of an omnipotent and perfectly good being is coherent. Therefore, God's existence is necessary. (N: God's existence is necessary; I: God's existence is impossible; C: God's existence is contingent; M: God's

existence is a matter of metaphysical luck; T: The concept of an omnipotent and perfectly good being is coherent)

9. Either the "eye for an eye" principle is interpreted literally, or it is interpreted figuratively. If it is interpreted literally, then the state must do to criminals what they have done to their victims. If the state must do to criminals what they have done to their victims, then the state must torture torturers. On the other hand, if the "eye for an eye" principle is interpreted figuratively, the state need only mete out punishments that are proportional to the crime. If the state need only mete out punishments that are proportional to the crime, then the state is free to give murderers life imprisonment rather than the death penalty. Now, the state must not torture torturers if such acts are immoral. And it is indeed immoral to torture torturers. Hence, the state is free to give murderers life imprisonment rather than the death penalty. (L: The "eye for an eye" principle is interpreted literally; F: The "eye for an eye" principle is interpreted figuratively; C: The state must do to criminals what they have done to their victims; T: The state must torture torturers; P: The state need only mete out punishments that are proportional to the crime; S: The state is free to give murderers life imprisonment rather than the death penalty; I: It is immoral to torture torturers)

10. Either Mary is in much pain or she isn't in much pain. And Mary lacks a capacity to make a rational decision about ending her life if she is in a lot of pain. On the other hand, given that Mary isn't in much pain, she is in no position to know what she will want when she is in much pain. Furthermore, Mary has no right to end her life if either she lacks a capacity to make a rational decision about ending her life or she is in no position to know what she will want when she is in much pain. But Mary has no right to "die with dignity" if she has no right to end her life. Therefore, Mary has no right to "die with dignity." (M: Mary is in much pain; R: Mary lacks a capacity to make a rational decision about ending her life; K: Mary is in no position to know what she will want when she is in much pain; E: Mary has no right to end her life; D: Mary has no right to "die with dignity")

8.2 Five Equivalence Rules

In developing a system of natural deduction, the logician is pulled in two directions. On the one hand, it is possible to develop a system with a small number of inference rules. But in systems with only a few rules, the proofs are often quite long and require much ingenuity. Moreover, the proof strategies tend to depart substantially from those employed in ordinary reasoning. On the other hand, it is possible to develop a system with a very large number of rules. Such systems allow for relatively short proofs, but most people find it difficult to remember a large number of rules. The present system of statement logic is a compromise that includes 20 rules altogether—8 implicational rules, 10 equivalence rules (introduced in this section and the next), conditional proof, and *reductio ad absurdum*.

Recall that two statements are *logically equivalent* if they validly imply each other. So, it is a valid inference to move from one statement to another that is logically equivalent to it. We can also say that two statements are logically equivalent if they agree in truth value regardless of the truth values assigned to their atomic components.[2] For example, $P \vee Q$ is logically equivalent to $Q \vee P$, and the inference from $P \vee Q$ to $Q \vee P$ is valid. Now, an equivalence rule, as the name suggests, is based on a logical equivalence. And our use of the equivalence rules depends on this further principle: *Within truth-functional logic, if we replace part of a compound statement with anything logically equivalent to that part, the resulting statement will have the same truth value as the original compound.* For example, if we start with $(P \vee Q) \rightarrow R$ and replace $(P \vee Q)$ with $(Q \vee P)$, we get a statement that has the same truth value as the first, namely, $(Q \vee P) \rightarrow R$. *And the inference from* $(P \vee Q) \rightarrow R$ *to* $(Q \vee P) \rightarrow R$ *is clearly valid because the two statements must have the same truth value.*

Five equivalence rules are introduced in this section and five more in the next section. Using the four-dot symbol (: :) to indicate logical equivalence, we can state our first equivalence rule, the rule of **double-negation**, as follows:

Rule 9: Double-negation (DN): $p :: {\sim}{\sim}p$

The four-dot symbol tells us that we may move validly from ${\sim}{\sim}p$ to p, as well as from p to ${\sim}{\sim}p$. All our equivalence rules are two-directional in this sense, unlike the implicational rules introduced in the previous section. For example, the rule of addition allows one to move from p to $p \vee q$, but it does not allow one to move from $p \vee q$ to p. Indeed, the latter move is invalid; here is a counterexample: "Either the number 3 is even, or the number 2 is even. So, the number 3 is even."

The rule of double-negation formalizes the intuition that any statement implies, and is implied by, the negation of its negation. Here are two English examples:

11. It is not true that Booth did not kill Lincoln. So, Booth killed Lincoln.
12. Booth killed Lincoln. So, it is not true that Booth did not kill Lincoln.

The usefulness of this rule is illustrated in constructing a proof for the following short argument:

13. If humans do not have free will, then they are not responsible for their actions. But obviously, humans are responsible for their actions. Thus, humans have free will. (F: Humans have free will; R: Humans are responsible for their actions)

Using the scheme of abbreviation provided, argument (13) translates into symbols as follows:

1. $\sim F \rightarrow \sim R$
2. R $\quad\therefore F$

The proof must include two applications of the double-negation rule:

3. $\sim\sim R$ 2, DN
4. $\sim\sim F$ 1, 3, MT
5. F 4, DN

Note that we cannot obtain *F* from the premises in one step by applying MT. MT tells us that if we have a conditional in one line of a proof and *the negation of the conditional's consequent* in another line of the proof, then we can infer the negation of the antecedent. But line (2) of the proof does not give us the negation of the consequent of line (1). The negation of $\sim R$ is $\sim\sim R$, and hence we must use the double-negation rule prior to applying MT.

As mentioned in the previous section, there is an important difference between implicational and equivalence rules as regards the construction of proofs. *We can apply equivalence rules to parts of lines in a proof and to entire lines.* We can do this because we never change the truth value of a statement by replacing some part of it with a logically equivalent expression. By contrast, *we can apply implicational rules only to entire lines in a proof*. The need for this restriction is illustrated by the following fallacious argument:

14. If Harry Truman was president in 1950, and Dwight Eisenhower was president in 1950, then America had two presidents in 1950. Therefore, if Truman was president in 1950, then America had two presidents in 1950. (T: Truman was president in 1950; E: Eisenhower was president in 1950; A: America had two presidents in 1950)

Without our restriction on implicational inference rules, we could construct the following proof:

1. $(T \cdot E) \rightarrow A$ $\quad\therefore T \rightarrow A$
2. $T \rightarrow A$ 1, *incorrect use of Simp* [not permitted]

Plainly, we do not want to allow this type of move. (An abbreviated truth table quickly reveals that the argument is invalid.) The proper use of both implicational and equivalence rules is illustrated in the following proof:

1. $(A \rightarrow B) \rightarrow (A \rightarrow \sim\sim C)$
2. A
3. $A \rightarrow D$
4. $D \rightarrow B$ $\quad\therefore C$

At this point, if we tried to apply MP to lines (1) and (2) to derive *B* or $\sim\sim C$, we would be misapplying MP. An implicational rule such as MP cannot be applied to a *part* of line (1); it must be applied to the whole line. So, we would need $A \rightarrow B$ to get $A \rightarrow {\sim}{\sim}C$ from line (1) by MP. On the other hand, since double-negation is an equivalence rule, we can, if we wish, apply double-negation to a part of a line. Thus, we can complete our proof as follows:

5. $(A \rightarrow B) \rightarrow (A \rightarrow C)$	1, DN
6. $A \rightarrow B$	3, 4, HS
7. $A \rightarrow C$	5, 6, MP
8. C	2, 7, MP

The fact that equivalence rules can be applied to parts of lines makes them very flexible tools to work with. But error will result if one fails to keep the distinction between implicational and equivalence rules firmly in mind. To repeat: The eight rules introduced in the previous section are all implicational rules (*modus ponens*, *modus tollens*, hypothetical syllogism, disjunctive syllogism, constructive dilemma, simplification, conjunction, and addition). The 10 rules introduced in this section and the next are all equivalence rules. For easy reference, a table of inference rules is provided on the inside front cover of this book.

Our second equivalence rule is **commutation**, which applies to both disjunctions and conjunctions:

Rule 10: Commutation (Com): $(p \vee q) :: (q \vee p)$
$(p \bullet q) :: (q \bullet p)$

Here are two English examples of commutation:

15. Either Sarah loves psychology, or Harlan hates history. So, either Harlan hates history, or Sarah loves psychology.
16. Frege is a logician, and Russell is a logician. So, Russell is a logician, and Frege is a logician.

The utility of the rule of commutation is revealed in constructing a proof for the following argument:

17. If pointless suffering occurs, then God is not both benevolent and omnipotent. But God is both omnipotent and benevolent. So, pointless suffering doesn't occur. (P: Pointless suffering occurs; B: God is benevolent; O: God is omnipotent)

1. $P \rightarrow {\sim}(B \bullet O)$	
2. $O \bullet B$	$\therefore {\sim}P$
3. $B \bullet O$	2, Com
4. ${\sim}{\sim}(B \bullet O)$	3, DN
5. ${\sim}P$	4, 1, MT

To underscore the difference between implicational and equivalence rules, it may be helpful to note that the following alternative proof is also correct:

3. $P \rightarrow \sim(O \bullet B)$	1, Com
4. $\sim\sim(O \bullet B)$	2, DN
5. $\sim P$	3, 4, MT

Here, the rule of commutation is applied to *part* of line (1) to obtain line (3).

The rule of **association** is so obvious that you may not have thought of it as involving an inference. It comes in two forms, one governing disjunctions and one governing conjunctions:

Rule 11: Association (As): $(p \vee (q \vee r)) :: ((p \vee q) \vee r)$
$(p \bullet (q \bullet r)) :: ((p \bullet q) \bullet r)$

In English, this sort of inference would normally be signaled by a shift in punctuation. Here is an example of the first form of association:

18. Either the alleged eyewitnesses of UFO landings are telling the truth, or they are lying or they've been duped. So, either the alleged eyewitnesses of UFO landings are telling the truth or they are lying, or they've been duped.

In our symbolic language, the parentheses play the role that the commas play in the English example. The practical value of the rule of association is illustrated in constructing a proof for the following short argument:

19. Either cigarette manufacturers are greedy or they are ignorant of cancer research, or they dislike young people. But it is not true that either cigarette manufacturers are ignorant of cancer research or they dislike young people. Therefore, cigarette manufacturers are greedy. (C: Cigarette manufacturers are greedy; R: Cigarette manufacturers are ignorant of cancer research; D: Cigarette manufacturers dislike young people)

1. $(C \vee R) \vee D$	
2. $\sim(R \vee D)$	$\therefore C$
3. $C \vee (R \vee D)$	1, As
4. C	2, 3, DS

Our next rule was first made explicit by the English logician Augustus De Morgan (1806–1871) and so is named after him. It comes in two forms. **De Morgan's laws** delineate the logical relations of negated conjunctions and negated disjunctions.

Rule 12: De Morgan's laws (DeM): $\sim(p \bullet q) :: (\sim p \vee \sim q)$
$\sim(p \vee q) :: (\sim p \bullet \sim q)$

Here is an English example of an inference endorsed by the first of De Morgan's laws:

20. Spot is not both a dog and a cat. So, either Spot is not a dog or Spot is not a cat.

The first law also tells us that we may reverse this reasoning and infer the *premise* of argument (20) from its conclusion. (This should make sense, as the premise and conclusion are logically equivalent.) Here is an English example of the second law:

21. It's not true that either hydrogen or oxygen is a metal. So, hydrogen is not a metal and oxygen is not a metal.

The second law also tells us that we may reverse this reasoning and infer the premise from the conclusion. As the following example illustrates, De Morgan's laws are quite useful in constructing proofs.

22. Either people are equal and deserve equal pay for equal work, or else people are not equal and do not deserve equal pay for equal work. People are not equal. So, people do not deserve equal pay for equal work. (E: People are equal; D: People deserve equal pay for equal work)

1. (E • D) ∨ (~E • ~D)	
2. ~E	∴ ~D
3. ~E ∨ ~D	2, Add
4. ~(E • D)	3, DeM
5. ~E • ~D	1, 4, DS
6. ~D	5, Simp

The strategy required in this proof is a bit indirect. The basic insight is that the second premise, $\sim E$, is clearly incompatible with the left disjunct of the first premise, $E \bullet D$. This means that an application of disjunctive syllogism is in the offing. But we have to use addition and one of De Morgan's laws before we can apply disjunctive syllogism.

Our next rule of inference relies on the logical equivalence between a conditional and its contrapositive. To form the contrapositive of a conditional, switch the antecedent and consequent and negate both. To illustrate, the contrapositive of "If Bob is an uncle, then Bob is male" is "If Bob is not male, then Bob is not an uncle." Let us call the inference rule itself **contraposition.**

Rule 13: Contraposition (Cont): $(p \rightarrow q) :: (\sim q \rightarrow \sim p)$

The utility of this rule becomes apparent in evaluating the following argument:

23. If it is wrong to use drugs only if they impair the user's mental functions, then it is not wrong to use caffeine. And if drugs do not impair the user's mental functions, then it is not wrong to use drugs. Hence, it is not wrong to use caffeine. (W: It is wrong to use drugs; D: Drugs impair the user's mental functions; C: It is wrong to use caffeine)

1. (W → D) → ~C	
2. ~D → ~W	∴ ~C

3. $W \rightarrow D$ 2, Cont
4. $\sim C$ 1, 3, MP

To emphasize the point that equivalence rules can be applied to *part* of a line, let us note that the proof could also be completed as follows:

3. $(\sim D \rightarrow \sim W) \rightarrow \sim C$ 1, Cont
4. $\sim C$ 2, 3, MP

Here, contraposition is applied to *part* of line (1) to obtain line (3).

The five rules introduced in this section may seem obvious or even trivial, but some logicians have rejected one or more of them. This results from skepticism about the **law of the excluded middle**, which says that for any given statement, either it or its negation is true. Using statement variables, we can state the law of the excluded middle as follows: $p \vee \sim p$. One group of logicians who reject the law of the excluded middle is the intuitionists. The intuitionists hold that the truth of a statement consists in there being a proof of it. Thus, to prove that any statement of the form $p \vee q$ is true, we must either prove that p is true or prove that q is true. Now, consider Goldbach's conjecture, which states that every even number greater than 2 is equal to the sum of two primes. No one has proved that this conjecture is true, and no one has proved that it is not true. Let us symbolize Goldbach's conjecture with the letter "G". According to the intuitionists, then, the statement $G \vee \sim G$ is not true because neither disjunct has been proved, and so the law of the excluded middle is not true.[3]

However, given the rules introduced in this section, we cannot deny the law of the excluded middle unless we are prepared to deny the **law of noncontradiction**, which states that contradictions are never true. Using statement variables, the law of noncontradiction can be expressed as follows: $\sim(p \bullet \sim p)$. And all logicians endorse the law of noncontradiction. Now, consider the following proof:

1. $\sim(G \bullet \sim G)$ $\therefore G \vee \sim G$
2. $\sim G \vee \sim\sim G$ 1, DeM
3. $\sim G \vee G$ 2, DN
4. $G \vee \sim G$ 3, Com

The premise says that Goldbach's conjecture is not both true and false (or that Goldbach's conjecture and its negation are not *both* true). And the conclusion says that Goldbach's conjecture is either true or false (or that either Goldbach's conjecture or its negation is true). It appears, then, that if we wish to reject the law of the excluded middle, we must also reject at least one of the following: De Morgan's laws, double-negation, commutation, or the law of noncontradiction. But each of these is very hard to deny.

Summary of the First Set of Equivalence Rules

9. Double-negation (DN):	$p :: \sim\sim p$
10. Commutation (Com):	$(p \vee q) :: (q \vee p)$ $(p \bullet q) :: (q \bullet p)$
11. Association (As):	$(p \vee (q \vee r)) :: ((p \vee q) \vee r)$ $(p \bullet (q \bullet r)) :: ((p \bullet q) \bullet r)$
12. De Morgan's laws (DeM):	$\sim(p \bullet q) :: (\sim p \vee \sim q)$ $\sim(p \vee q) :: (\sim p \bullet \sim q)$
13. Contraposition (Cont):	$(p \rightarrow q) :: (\sim q \rightarrow \sim p)$

As you complete the exercises that follow, keep in mind that the three rules of thumb provided in the previous section still apply: Start with the conclusion and work backward; break premises down into simpler components using MP, MT, Simp, DS, and so on; and if a "new" statement letter appears in the conclusion, use addition. To these three rules of thumb, we now add the following:

Rule of Thumb 4: It is often useful to consider logically equivalent forms of the conclusion.

For example, suppose the conclusion is $\sim(A \bullet B)$. Then it may help to notice that the conclusion is logically equivalent to $\sim A \vee \sim B$, according to De Morgan's laws. Or if the conclusion is $\sim D \rightarrow \sim C$, it may be helpful to note that the conclusion is equivalent to $C \rightarrow D$, by contraposition.

Rule of Thumb 5: Both conjunction and addition can lead to useful applications of De Morgan's laws.

Consider the following examples:

1. $\sim E$			1. $\sim G$	
2. $\sim F$			2. $\sim G \vee \sim H$	1, Add
3. $\sim E \bullet \sim F$	1, 2, Conj		3. $\sim(G \bullet H)$	2, DeM
4. $\sim(E \vee F)$	3, DeM			

As before, rules of thumb are to be taken as helpful hints. They do not automatically provide a solution in every case. A summary of rules of thumb for constructing proofs appears in section 8.5 on page 349.

Exercise 8.2

Part A: Annotating Annotate the following short proofs. (In each case, the argument has only one premise.)

* **1.** 1. $\sim\sim A \rightarrow B \quad \therefore A \rightarrow B$
2. $A \rightarrow B$

2. 1. $\sim C \rightarrow \sim D \quad \therefore D \rightarrow C$
2. $D \rightarrow C$

3. 1. $\sim(E \bullet \sim D) \quad \therefore D \vee \sim E$
2. $\sim E \vee \sim\sim D$
3. $\sim E \vee D$
4. $D \vee \sim E$

* **4.** 1. $\sim(E \vee D) \quad \therefore \sim D$
2. $\sim E \bullet \sim D$
3. $\sim D$

5. 1. $\sim A \bullet [(A \vee B) \vee C] \quad \therefore B \vee C$
2. $\sim A$
3. $(A \vee B) \vee C$
4. $A \vee (B \vee C)$
5. $B \vee C$

6. 1. $F \bullet (G \bullet R) \quad \therefore G \bullet F$
2. $(F \bullet G) \bullet R$
3. $F \bullet G$
4. $G \bullet F$

* **7.** 1. $[(P \rightarrow Q) \rightarrow R] \bullet (\sim Q \rightarrow \sim P) \quad \therefore \sim\sim R$
2. $\sim Q \rightarrow \sim P$
3. $P \rightarrow Q$
4. $(P \rightarrow Q) \rightarrow R$
5. R
6. $\sim\sim R$

8. 1. $[\sim(S \bullet T) \vee \sim\sim U] \bullet (T \bullet S) \quad \therefore U$
2. $T \bullet S$
3. $S \bullet T$
4. $\sim(S \bullet T) \vee \sim\sim U$
5. $\sim\sim(S \bullet T)$
6. $\sim\sim U$
7. U

9. 1. $\sim W \vee (\sim X \vee \sim Y) \quad \therefore \sim Y \vee \sim(W \bullet X)$
2. $(\sim W \vee \sim X) \vee \sim Y$
3. $\sim(W \bullet X) \vee \sim Y$
4. $\sim Y \vee \sim(W \bullet X)$

* 10. 1. $[\sim O \rightarrow (\sim M \rightarrow \sim N)] \cdot \sim(N \rightarrow M)$ $\therefore O$
2. $\sim O \rightarrow (\sim M \rightarrow \sim N)$
3. $\sim(N \rightarrow M)$
4. $\sim(\sim M \rightarrow \sim N)$
5. $\sim\sim O$
6. O

11. 1. $\sim\sim P \cdot \sim P$ $\therefore W$
2. $\sim\sim P$
3. $\sim\sim P \vee W$
4. $P \vee W$
5. $\sim P$
6. W

12. 1. $\sim T$ $\therefore \sim(T \cdot \sim\sim S)$
2. $\sim T \vee \sim S$
3. $\sim(T \cdot S)$
4. $\sim(T \cdot \sim\sim S)$

* 13. 1. $\sim A$ $\therefore \sim[(B \cdot C) \cdot A]$
2. $\sim(B \cdot C) \vee \sim A$
3. $\sim[(B \cdot C) \cdot A]$

14. 1. $(S \rightarrow G) \cdot (G \rightarrow T)$ $\therefore \sim T \rightarrow \sim S$
2. $S \rightarrow G$
3. $G \rightarrow T$
4. $S \rightarrow T$
5. $\sim T \rightarrow \sim S$

15. 1. $\sim R \rightarrow \sim S$ $\therefore \sim P \vee (S \rightarrow \sim\sim R)$
2. $S \rightarrow R$
3. $S \rightarrow \sim\sim R$
4. $\sim P \vee (S \rightarrow \sim\sim R)$

Part B: Correct or Incorrect? Some of the following inferences are correct applications of our rules, and some are not. If an inference is a correct application of our rules, name the rule. If an inference is not a correct application of our rules, explain why it is not. (The question is whether the conclusion in each case can be reached *in a single step* from the premise(s) by an application of one of our rules.)

* 1. $\sim(\sim E \vee B)$
$\therefore \sim\sim E \cdot \sim B$

2. $\sim B \rightarrow \sim C$
$\therefore C \rightarrow B$

3. $\sim(F \vee G)$
$\therefore \sim F \vee \sim G$

* 4. $\sim W \vee \sim Z$
$\therefore \sim(W \cdot Z)$

5. $A \cdot \sim B$
$\therefore \sim B \cdot A$

6. $\sim D \rightarrow \sim E$
$\therefore \sim\sim E \rightarrow \sim\sim D$

* 7. $\sim S \vee T$
$\therefore \sim(S \cdot \sim T)$

8. $\sim J \vee \sim\sim K$
$\therefore \sim(J \cdot \sim K)$

9. $P \to {\sim}Q$
$\therefore Q \to {\sim}P$

* 10. $O \to R$
$\therefore {\sim}R \to {\sim}O$

11. $[B \vee (C \vee A)] \leftrightarrow D$
$\therefore [(C \vee B) \vee A] \leftrightarrow D$

12. ${\sim}(D \cdot C) \to E$
$\therefore ({\sim}D \vee {\sim}C) \to E$

* 13. ${\sim}(L \cdot {\sim}M)$
$\therefore {\sim}({\sim}M \cdot L)$

14. ${\sim}(U \cdot {\sim}Z)$
$\therefore {\sim}U \vee {\sim}{\sim}Z$

15. $({\sim}{\sim}N \vee {\sim}M) \leftrightarrow (L \cdot K)$
$\therefore {\sim}({\sim}N \cdot M) \leftrightarrow (L \cdot K)$

* 16. ${\sim}[(O \cdot {\sim}P) \cdot W]$
$\therefore {\sim}[O \cdot ({\sim}P \cdot W)]$

17. ${\sim}(R \vee {\sim}Q)$
$\therefore {\sim}R \cdot {\sim}{\sim}Q$

18. ${\sim}{\sim}S \leftrightarrow T$
$\therefore S \leftrightarrow T$

* 19. ${\sim}{\sim}(U \vee W)$
$\therefore {\sim}({\sim}U \cdot {\sim}W)$

20. ${\sim}(X \to Y)$
$\therefore {\sim}X \to {\sim}Y$

Part C: Proofs Construct proofs for each of the following symbolic arguments. Commas are used to mark the breaks between premises. (Each proof can be completed in fewer than 10 steps, including premises.)

* 1. ${\sim}(C \cdot D),\ {\sim}C \to S,\ {\sim}D \to T \ \therefore S \vee T$

2. $(W \to U) \cdot {\sim}X \ \therefore {\sim}U \to {\sim}W$

3. $F \to {\sim}G,\ G \ \therefore {\sim}F$

* 4. ${\sim}({\sim}A \vee B) \ \therefore A$

5. $({\sim}P \to Q) \cdot {\sim}Q \ \therefore P$

6. ${\sim}(N \vee M),\ {\sim}L \to (M \vee N) \ \therefore L$

* 7. $(A \vee B) \vee C,\ {\sim}A \ \therefore C \vee B$

8. $(W \cdot {\sim}X) \vee (Y \cdot Z),\ ({\sim}X \cdot W) \to U,\ (Y \cdot Z) \to T \ \therefore U \vee T$

9. ${\sim}(S \vee R),\ P \to R \ \therefore {\sim}P$

* 10. $F \to (G \cdot H),\ (H \cdot G) \to J \ \therefore F \to J$

11. $K \vee (L \vee S),\ {\sim}(K \vee L) \ \therefore S$

12. ${\sim}P,\ {\sim}(P \vee Q) \to {\sim}R,\ {\sim}Q \ \therefore {\sim}R$

* 13. ${\sim}S \to (T \cdot U),\ ({\sim}S \to X) \to {\sim}Z,\ (U \cdot T) \to X \ \therefore {\sim}Z$

14. ${\sim}({\sim}B \to A),\ C \to ({\sim}A \to B) \ \therefore {\sim}C$

15. ${\sim}E,\ F \to (D \vee E),\ {\sim}D \ \therefore {\sim}F$

* 16. $(K \vee P) \vee X,\ K \to {\sim}O,\ (P \vee X) \to {\sim}L \ \therefore {\sim}(O \cdot L)$

17. $(G \vee H) \to (J \vee K) \ \therefore {\sim}(J \vee K) \to {\sim}(H \vee G)$

18. ${\sim}A \to {\sim}{\sim}R,\ G \to {\sim}U,\ {\sim}A \vee G \ \therefore {\sim}({\sim}R \cdot U)$

* 19. ${\sim}(L \cdot M) \to {\sim}(N \vee O) \ \therefore (O \vee N) \to (M \cdot L)$

20. $B \to E,\ (B \vee C) \vee D,\ (D \vee C) \to F \ \therefore E \vee F$

21. $W \rightarrow \sim U,\ W \vee X,\ \sim T \rightarrow (Z \bullet U),\ X \rightarrow \sim Z\ \therefore T$

*22. $\sim(\sim P \bullet Q),\ \sim Q \rightarrow R,\ P \rightarrow \sim S\ \therefore R \vee \sim S$

23. $\sim B,\ A \rightarrow (B \bullet C)\ \therefore \sim A$

24. $[S \rightarrow (J \bullet Q)] \bullet \sim Q\ \therefore \sim S$

*25. $\sim B,\ \sim(C \bullet B) \rightarrow C,\ \sim F \rightarrow \sim C\ \therefore F$

Part D: Longer Proofs Construct proofs to show that the following arguments are valid. Commas are used to mark the breaks between premises.

* 1. $\sim\sim T \vee \sim R,\ \sim(S \vee \sim R),\ (T \bullet \sim S) \rightarrow \sim Q,\ W \rightarrow Q\ \therefore \sim W$

2. $\sim(J \bullet L),\ (\sim J \vee \sim L) \rightarrow \sim M,\ \sim E \vee (M \vee \sim S)\ \therefore \sim(S \bullet E)$

3. $E \rightarrow [\sim(H \vee K) \rightarrow R],\ \sim\sim E \bullet (\sim H \bullet \sim K)\ \therefore \sim\sim R$

* 4. $B \rightarrow E,\ \sim F \vee G,\ (B \bullet C) \bullet D,\ (D \bullet C) \rightarrow F\ \therefore E \bullet G$

5. $P \vee (Q \vee R),\ (Q \vee P) \rightarrow \sim S,\ R \rightarrow \sim T,\ U \rightarrow (S \bullet T)\ \therefore \sim U \vee Z$

6. $\sim\sim W \bullet [(X \vee W) \rightarrow Y],\ H \rightarrow \sim Y\ \therefore \sim H$

* 7. $\sim(B \bullet \sim C),\ \sim B \rightarrow D,\ C \rightarrow \sim E\ \therefore \sim E \vee D$

8. $(F \bullet G) \rightarrow (H \bullet J),\ (J \bullet H) \rightarrow (K \vee L),\ (L \vee K) \rightarrow M\ \therefore (G \bullet F) \rightarrow M$

9. $\sim Y \vee N,\ (Y \bullet \sim N) \vee (Y \bullet Z),\ (Z \bullet Y) \rightarrow \sim\sim U\ \therefore U \vee \sim V$

*10. $\sim A \rightarrow \sim B,\ D \rightarrow E,\ (B \rightarrow A) \rightarrow (C \vee D),\ C \rightarrow F\ \therefore E \vee F$

11. $\sim(H \leftrightarrow G) \vee \sim J,\ K \rightarrow (H \leftrightarrow G),\ \sim L \rightarrow J\ \therefore \sim(K \bullet \sim L)$

12. $(X \bullet Q) \rightarrow (Z \bullet \sim T),\ R \bullet (T \vee \sim Z)\ \therefore (\sim X \vee \sim Q) \bullet R$

*13. $\sim[(M \vee N) \vee O],\ (P \bullet R) \rightarrow N,\ \sim P \rightarrow T,\ \sim R \rightarrow S\ \therefore T \vee S$

14. $Z \rightarrow (U \bullet X),\ \sim[(U \bullet W) \bullet X],\ W\ \therefore \sim Z$

15. $\sim(\sim A \bullet B) \leftrightarrow \sim(C \vee \sim D)\ \therefore (\sim B \vee A) \leftrightarrow (D \bullet \sim C)$

*16. $\sim[(E \bullet F) \vee G],\ (H \vee \sim E) \rightarrow G\ \therefore \sim(F \vee H)$

17. $\sim(R \rightarrow S) \rightarrow \sim(\sim T \rightarrow \sim U),\ \sim W \rightarrow T,\ U \rightarrow \sim W,\ \sim S\ \therefore \sim R$

18. $\sim[(L \vee M) \bullet N],\ (P \rightarrow \sim Q) \rightarrow N,\ Q \rightarrow \sim P\ \therefore \sim M$

*19. $[(A \bullet B) \vee \sim C] \rightarrow (\sim X \bullet \sim Y),\ \sim(Y \vee X) \rightarrow Z,\ \sim C \vee (A \bullet B)$
$\therefore \sim\sim Z$

20. $H \vee G,\ \sim(\sim D \rightarrow E),\ (F \vee G) \rightarrow (\sim E \rightarrow D)\ \therefore \sim J \vee H$

21. $\sim(\sim P \bullet \sim Q),\ Q \rightarrow (X \vee R),\ P \rightarrow Y\ \therefore X \vee (Y \vee R)$

*22. $A \vee (B \vee C),\ \sim A \bullet \sim C\ \therefore [(B \vee C) \bullet \sim(A \vee C)] \bullet (A \vee B)$

23. $\sim F,\ \sim(F \bullet \sim S) \rightarrow \sim P,\ (\sim S \bullet F) \vee \sim T\ \therefore \sim(P \vee T)$

24. $O \rightarrow (H \bullet M),\ (O \rightarrow G) \rightarrow (H \rightarrow \sim M),\ \sim G \rightarrow (\sim H \vee \sim M)$
$\therefore H \rightarrow \sim M$

*25. $Z \bullet Y,\ T \rightarrow X,\ \sim Y \rightarrow \sim S,\ \sim(X \vee Y) \vee \sim Z\ \therefore \sim(T \vee S) \bullet Y$

8.3 Five More Equivalence Rules

To this point, our system of natural deduction includes 8 implicational rules and 5 equivalence rules. With these 13 rules, we can construct proofs for many valid arguments in statement logic. But we need 5 more equivalence rules (plus a rule called "conditional proof") if our system of natural deduction is to be able to *prove valid* every argument that is valid according to the truth table method. So, in this section, we add 5 more equivalence rules to our system.

The rule of **distribution** tells us how certain combinations of the dot and the vee interrelate. It comes in two forms.

Rule 14: Distribution (Dist): $(p \cdot (q \vee r)) :: ((p \cdot q) \vee (p \cdot r))$
$(p \vee (q \cdot r)) :: ((p \vee q) \cdot (p \vee r))$

To grasp these inferences, think about them truth functionally. For example, consider the first form of distribution: Suppose $(p \cdot (q \vee r))$ is true; then p is true and $(q \vee r)$ is true; so either $(p \cdot q)$ is true or $(p \cdot r)$ is true (or both). Similarly, suppose $((p \cdot q) \vee (p \cdot r))$ is true. If $(p \cdot q)$ is true, then $(p \cdot (q \vee r))$ must be true, too; but if $(p \cdot r)$ is true, then, again, $(p \cdot (q \vee r))$ must be true. Notice that when distribution is applied correctly, the main logical operator changes (either from the dot to the vee or from the vee to the dot). Here are some English examples of distribution:

24. "Bats are animals, and they are either mammals or birds" implies (and is implied by) "Either bats are animals and mammals, or bats are animals and birds."
25. "Either Bill lost the lottery, or Bill won and he is rich" implies (and is implied by) "Either Bill lost the lottery or he won, and either Bill lost the lottery or he is rich."

The utility of the rule of distribution is brought out when we construct a proof of the following argument:

26. Either Fiona is insane, or she is guilty and a liar. But if Fiona is either insane or a liar, then she is dangerous. It follows that Fiona is dangerous. (F: Fiona is insane; G: Fiona is guilty; L: Fiona is a liar; D: Fiona is dangerous)

1. $F \vee (G \cdot L)$	
2. $(F \vee L) \rightarrow D$	$\therefore$ D
3. $(F \vee G) \cdot (F \vee L)$	1, Dist
4. $F \vee L$	3, Simp
5. D	2, 4, MP

Perhaps because distribution appears a bit complex, there is some tendency to overlook occasions for its use when constructing proofs, but it is often quite useful.

The rule of **exportation** tells us that statements of the form "If p and q, then r" are logically equivalent to statements of the form "If p, then if q, then r." In symbols, we have the following:

Rule 15: Exportation (Ex): $((p \bullet q) \rightarrow r)$: : $(p \rightarrow (q \rightarrow r))$

Here is an English example:

27. "If Sue is intelligent and she studies hard, then she gets good grades" implies (and is implied by) "If Sue is intelligent, then if she studies hard, she gets good grades."

A proof of the following argument will illustrate a typical usage of exportation.

28. If World War I was not a war in defense of the U.S.A., and only wars of defense are just, then the American participation in World War I was not just. World War I was not a war in defense of the U.S.A. It follows that if only wars of defense are just, then the American participation in World War I was not just. (W: World War I was a war in defense of the U.S.A.; D: Only wars of defense are just; J: American participation in World War I was just)

1. $(\sim W \bullet D) \rightarrow \sim J$	
2. $\sim W$	$\therefore D \rightarrow \sim J$
3. $\sim W \rightarrow (D \rightarrow \sim J)$	1, Ex
4. $D \rightarrow \sim J$	2, 3, MP

The **redundancy** rule is obviously valid, and as the name suggests, it allows us to eliminate certain types of redundancy.

Rule 16: Redundancy (Re): p : : $(p \bullet p)$
p : : $(p \vee p)$

A proof of the following argument reveals a typical use of this rule.

29. Either pain is real or it is an illusion. If pain is real, then pain is bad. And if pain is an illusion, then pain is bad. Accordingly, pain is bad. (R: Pain is real; I: Pain is an illusion; B: Pain is bad)

1. $R \vee I$	
2. $R \rightarrow B$	
3. $I \rightarrow B$	$\therefore B$
4. $B \vee B$	1, 2, 3, CD
5. B	4, Re

Note that the rule allows us to introduce redundancy as well as to eliminate it. For example, the redundancy rule allows us to move from $\sim A$ to $\sim A \bullet \sim A$ and from R to $R \vee R$.

The rule of **material equivalence** gives us a way of handling biconditionals. It comes in two forms. The first form tells us that a biconditional is logically equivalent to a *conjunction* of two conditionals. The second form tells us that a biconditional is logically equivalent to a *disjunction* of two conjunctions. The second form makes sense if you remember the truth table for the biconditional:

$(p \leftrightarrow q)$ is true if either p and q are both true or p and q are both false; otherwise, $(p \leftrightarrow q)$ is false.

Rule 17: Material equivalence (ME): $(p \leftrightarrow q) :: ((p \rightarrow q) \cdot (q \rightarrow p))$
$(p \leftrightarrow q) :: ((p \cdot q) \vee (\sim p \cdot \sim q))$

A proof of the following argument will illustrate a typical usage of material equivalence.

30. Withholding medical treatment is wrong if and only if either the patient has a valuable future life or the family insists on medical treatment. But the patient is brain dead. And if the patient is brain dead, then he has not got a valuable future life. Furthermore, it is not the case that the family insists on medical treatment. It follows that withholding medical treatment is not wrong. (W: Withholding medical treatment is wrong; L: The patient has a valuable future life; F: The family insists on medical treatment; B: The patient is brain dead)

1. $W \leftrightarrow (L \vee F)$	
2. B	
3. $B \rightarrow \sim L$	
4. $\sim F$	$\therefore \sim W$
5. $\sim L$	2, 3, MP
6. $[W \rightarrow (L \vee F)] \cdot [(L \vee F) \rightarrow W]$	1, ME
7. $W \rightarrow (L \vee F)$	6, Simp
8. $\sim L \cdot \sim F$	5, 4, Conj
9. $\sim(L \vee F)$	8, DeM
10. $\sim W$	7, 9, MT

The last of our equivalence rules is called **material implication**. It is based on the logical equivalence between statements of the form $(p \rightarrow q)$ and a *disjunction* whose disjuncts are the consequent of the conditional and the negation of its antecedent. This equivalence can easily be demonstrated with a truth table.

Rule 18: Material implication (MI): $(p \rightarrow q) :: (\sim p \vee q)$

Without material implication, our proof system would lack the capacity to prove valid every argument that is valid according to the truth table method. But it is important to remember that $\sim p \vee q$ and $p \rightarrow q$ are equivalent because of the truth-functional definition we have given the arrow. As we saw in Chapter 7, not every English statement of the form "If p, then q" is equivalent to "Either not p or q." For example, "If the Eiffel Tower is in Ohio, then it is in France" is intuitively false; but the disjunction "Either the Eiffel Tower is not in Ohio, or the Eiffel Tower is in France" is true, since the Eiffel Tower is in fact in France (and not in Ohio). Accordingly, we include the rule of material implication in our system, but with the realization that if a proof relies on this rule, it may fail to mirror our intuitive logical convictions about English conditionals.

Our proof of the following argument makes a strategic use of both material implication and the rule of distribution.

31. If either humans do not need meat or eating meat is unhealthy, then humans should not eat meat. Hence, if humans do not need meat, then humans should not eat meat. (N: Humans need meat; E: Eating meat is unhealthy; S: Humans should eat meat)

1. $(\sim N \vee E) \rightarrow \sim S$	$\therefore \sim N \rightarrow \sim S$
2. $\sim(\sim N \vee E) \vee \sim S$	1, MI
3. $\sim S \vee \sim(\sim N \vee E)$	2, Com
4. $\sim S \vee (\sim\sim N \bullet \sim E)$	3, DeM
5. $(\sim S \vee \sim\sim N) \bullet (\sim S \vee \sim E)$	4, Dist
6. $\sim S \vee \sim\sim N$	5, Simp
7. $\sim\sim N \vee \sim S$	6, Com
8. $\sim N \rightarrow \sim S$	7, MI

This proof is rather complex, and it suggests the following rules of thumb (to be added to the five rules of thumb introduced previously):

Rule of Thumb 6: Material implication can lead to useful applications of distribution.

This is illustrated by lines (2) through (5) in the previous proof. But here is a simpler case:

1. $A \rightarrow (B \bullet C)$	
2. $\sim A \vee (B \bullet C)$	1, MI
3. $(\sim A \vee B) \bullet (\sim A \vee C)$	2, Dist

Rule of Thumb 7: Distribution can lead to useful applications of simplification.

This rule of thumb is illustrated in lines (4) through (6) in the previous proof, but here is another example:

1. $(D \bullet E) \vee (D \bullet F)$	
2. $D \bullet (E \vee F)$	1, Dist
3. D	2, Simp

At least one more rule of thumb (not suggested by the previous proof) may be helpful as you complete the exercises at the end of this section:

Rule of Thumb 8: Addition can lead to useful applications of material implication.

Here are two examples:

Summary of the Second Set of Equivalence Rules	
14. Distribution (Dist):	$(p \bullet (q \vee r)) :: ((p \bullet q) \vee (p \bullet r))$
	$(p \vee (q \bullet r)) :: ((p \vee q) \bullet (p \vee r))$
15. Exportation (Ex):	$((p \bullet q) \rightarrow r) :: (p \rightarrow (q \rightarrow r))$
16. Redundancy (Re):	$p :: (p \bullet p)$
	$p :: (p \vee p)$
17. Material equivalence (ME):	$(p \leftrightarrow q) :: ((p \rightarrow q) \bullet (q \rightarrow p))$
	$(p \leftrightarrow q) :: ((p \bullet q) \vee (\sim p \bullet \sim q))$
18. Material implication (MI):	$(p \rightarrow q) :: (\sim p \vee q)$

1. B		1. ~F	
2. ~A ∨ B	1, Add	2. ~F ∨ G	1, Add
3. A → B	2, MI	3. F → G	2, MI

A summary of rules of thumb for constructing proofs is provided in section 8.5 on page 349.

In closing this section, let us reflect briefly on the value of proofs. What good are they? First, many valid arguments are sufficiently complex to dazzle one's logical intuitions. In such cases, our proof system comes into its own by enabling us to show how we can get from the premises to the conclusion *using only the rules we have explicitly adopted.* So, unless we have doubts about our system of rules, a proof should settle all doubts about the validity of even very complicated arguments. Second, suppose you claim that an argument is valid and someone else claims that it isn't. What can you do? Well, if the argument can be shown to be valid by means of a proof, then this should settle the matter (unless the other person rejects one or more of the rules in our system). The power of logic consists partly in the fact that in so many cases, it can settle the question of an argument's validity. And once we determine that an argument is valid, the question of its soundness turns entirely on whether its premises are true.

Exercise 8.3

Part A: Annotating Annotate the following short proofs. (In each case, the argument has only one premise.)

* 1. 1. B ↔ E ∴ E → B
2. (B → E) • (E → B)
3. E → B

2. 1. (B • C) ∨ (~B • ~C) ∴ B ↔ C
2. B ↔ C

3. 1. $\sim(A \bullet A) \vee (B \vee B)$ $\therefore A \to B$
   2. $\sim(A \bullet A) \vee B$
   3. $(A \bullet A) \to B$
   4. $A \to B$

* 4. 1. $H \to (J \to \sim H)$ $\therefore H \to \sim J$
   2. $H \to (\sim\sim H \to \sim J)$
   3. $(H \bullet \sim\sim H) \to \sim J$
   4. $(H \bullet H) \to \sim J$
   5. $H \to \sim J$

5. 1. $P \bullet \sim Q$ $\therefore P \bullet (\sim Q \vee R)$
   2. $(P \bullet \sim Q) \vee (P \bullet R)$
   3. $P \bullet (\sim Q \vee R)$

6. 1. $F \vee (\sim G \bullet H)$ $\therefore G \to F$
   2. $(F \vee \sim G) \bullet (F \vee H)$
   3. $F \vee \sim G$
   4. $\sim G \vee F$
   5. $G \to F$

* 7. 1. $M \to \sim N$ $\therefore N \to \sim M$
   2. $\sim M \vee \sim N$
   3. $\sim N \vee \sim M$
   4. $N \to \sim M$

8. 1. $\sim S \leftrightarrow T$ $\therefore (\sim S \bullet T) \vee (S \bullet \sim T)$
   2. $(\sim S \bullet T) \vee (\sim\sim S \bullet \sim T)$
   3. $(\sim S \bullet T) \vee (S \bullet \sim T)$

9. 1. $(B \bullet B) \vee (C \bullet D)$ $\therefore B \vee D$
   2. $B \vee (C \bullet D)$
   3. $(B \vee C) \bullet (B \vee D)$
   4. $B \vee D$

* 10. 1. $(U \to U) \vee (\sim U \to U)$ $\therefore \sim U \vee U$
   2. $(\sim U \vee U) \vee (\sim U \to U)$
   3. $(\sim U \vee U) \vee (\sim\sim U \vee U)$
   4. $(\sim U \vee U) \vee (U \vee U)$
   5. $(\sim U \vee U) \vee U$
   6. $\sim U \vee (U \vee U)$
   7. $\sim U \vee U$

11. 1. $\sim(T \vee T)$ $\therefore T \to \sim T$
   2. $\sim T \bullet \sim T$
   3. $\sim T$
   4. $\sim T \vee \sim T$
   5. $T \to \sim T$