

## Logical Equivalence

Logical equivalence is the relationship between any two (or more) propositions that always have the same truth value. For any two equivalent propositions, if one of them is true then so is the other, and if one of them is false then so is the other. Logical equivalence helps us to explain how conjunctions and disjunctions exhibit the property of commutativity. Conjunctions are commutative, because the order of the conjuncts doesn't influence the truth value of the conjunction. Consider the truth table below.

<b>P</b>	<b>Q</b>	<b><math>P \wedge Q</math></b>	<b><math>Q \wedge P</math></b>
TRUE	TRUE	TRUE	TRUE
FALSE	FALSE	FALSE	FALSE
TRUE	FALSE	FALSE	FALSE
FALSE	TRUE	FALSE	FALSE

Just like conjunctions, disjunctions are commutative, because the order of the disjuncts doesn't influence the truth value of the disjunction. Consider the truth table below.

<b>P</b>	<b>Q</b>	<b><math>P \vee Q</math></b>	<b><math>Q \vee P</math></b>
TRUE	TRUE	TRUE	TRUE
FALSE	FALSE	FALSE	FALSE
TRUE	FALSE	TRUE	TRUE
FALSE	TRUE	TRUE	TRUE

To understand the logical consequences of logical equivalence, let us consider two interesting cases: *double negation elimination* and *De Morgan's Laws*.

### Double Negation Elimination

Double negation elimination is the rule that, whenever a proposition is negated more than once, an even number of negations can be eliminated without changing the truth-value. As Leibniz would have said, any even number of negations can be eliminated *salva veritate*, because the truth value would be preserved. Consider the truth-table below.

<b>P</b>	<b>~P</b>	<b>~~P</b>	<b>~~~P</b>
TRUE	FALSE	TRUE	FALSE
FALSE	TRUE	FALSE	TRUE

In the truth table, propositions “P” and “~P” are contradictories. Just like how the contradictory of “P” is “~P”, the contradictory of “~P” is “~(~P)”, and “~(~P)” is the same as “~~P”. Accordingly, “~P” and “~~P” are contradictories. Just like how the contradictory of “~P” is “~~P”, the contradictory of “~~P” is “~(~~P)”, and “~(~~P)” is the same as “~~~P”. Accordingly, “~~P” and “~~~P” are contradictories.

Double negation elimination implies logical equivalence. Consider “P” and “~~P”. If one is true then so is the other, and if one is false then so is the other. Likewise, consider “~P” and “~~~P”. If one is true then so is the other, and if one is false then so is the other. By negating the contradictory of a statement, the negated contradictory is equivalent to the original statement. For example, “P” and “~P” are contradictories, so negating “~P” results in “~~P”, which is equivalent to the original statement “P”. Likewise, “~P” and “~~P” are contradictories, so negating “~~P” results in “~~~P”, which is equivalent to the original statement “~P”.

### De Morgan's Laws

De Morgan's Laws are two laws about logical equivalence involving conjunctions, disjunctions, and negation. According to De Morgan's Laws, the statement " $\sim(P \wedge Q)$ " is equivalent to the statement " $\sim P \vee \sim Q$ ", and the statement " $\sim(P \vee Q)$ " is equivalent to the statement " $\sim P \wedge \sim Q$ ". To see why each pair of statements is a pair of *equivalent* statements, fill out the truth table below.

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$P \vee Q$	$\sim(P \wedge Q)$	$\sim(P \vee Q)$	$\sim P \vee \sim Q$	$\sim P \wedge \sim Q$
TRUE	TRUE								
FALSE	FALSE								
TRUE	FALSE								
FALSE	TRUE								

Consider statements " $\sim(P \wedge Q)$ " and " $\sim P \vee \sim Q$ ". If one of them is true then the other one is false, and if one of them is false then the other one is true. Therefore, they are *equivalent* statements. Likewise, consider statements " $\sim(P \vee Q)$ " and " $\sim P \wedge \sim Q$ ". If one of them is true then the other one is false, and if one of them is false then the other one is true. Therefore, they are *equivalent* statements.

## Questions

For 88 of the 100 points for this assignment, complete the following truth table.

P	$\sim P$	$P \wedge \sim P$	$P \vee \sim P$	$\sim(P \wedge \sim P)$	$\sim(P \vee \sim P)$	$\sim P \vee \sim \sim P$	$\sim P \wedge \sim \sim P$	$\sim P \vee P$	$\sim P \wedge P$	$P \vee \sim P$	$P \wedge \sim P$
TRUE											
FALSE											

For the remaining 12 of the 100 points for this assignment, answer the following three questions. (Each question is worth 4 points.)

- What rule is exhibited by the equivalence of " $\sim P \vee \sim \sim P$ " and " $\sim P \vee P$ "? Likewise, what rule is exhibited by the equivalence of " $\sim P \wedge \sim \sim P$ " and " $\sim P \wedge P$ "?
- In the truth table, locate the column for statement " $P \wedge \sim P$ ".
  - Which statement is the *contradictory* of " $P \wedge \sim P$ "?
  - Now that you have found the contradictory of " $P \wedge \sim P$ ", which statement is *equivalent* to the contradictory of " $P \wedge \sim P$ "?
- In the truth table, locate the column for statement " $P \vee \sim P$ ".
  - Which statement is the *contradictory* of " $P \vee \sim P$ "?
  - Now that you have found the contradictory of " $P \vee \sim P$ ", which statement is *equivalent* to the contradictory of " $P \vee \sim P$ "?