AP Precalculus - M3Y & M3Z

Polynomials - Homework 3

1. Solve the following inequalities:

   (i) $6x^2 - 30x + 70 \geq 3x^2 + 3x - 20$

   (ii) $x^3 + 2x^2 - 3x < 0$

   (iii) $x^4 - 13x^2 + 36 > 0$

   (iv) $(x^2 - 15x + 56)(-x^2 - 2x + 24)(x^2 - 5x + 9) \geq 0$

   (v) $(x+8)^2(x+5)(x+7)^3 > 0$

   (vi) $x^4 + 4x^3 - 12x^2 \leq 0$

2. Divide the following polynomials. Show your work and write the polynomial long division, as well as the degree of the remainder (or that the remainder is zero):

   (i) $4x^2 - 10x + 6$ by $4x + 2$

   (ii) $2x^5 - 3x^4 - x^2 + x + 4$ by $x^2 + 1$

   (iii) $x^5 + 3x + 2$ by $x^3 + 2x + 1$

   (iv) $-3x^5 + 4x^3 + 3x^2 + 12x - 10$ by $x^2 + 2x - 1$

   (v) $x^4 + 1$ by $x - 1$

   (vi) $x^5 + 1$ by $x^6 - 1$

3. In this problem, you are going to prove the following result:
   If $f(x)$ is a polynomial and $k \in \mathbb{R}$, then:

   $$x - k \text{ divides } f(x) \iff f(k) = 0 \text{ (that is, } k \text{ is a root of } f(x))$$

   (i) Using polynomials $q(x)$ and $r(x)$ to denote the quotient and the remainder, respectively, write the polynomial long division of $f(x)$ by $x - k$

   For ($\Longrightarrow$), assume that $x - k$ divides $f(x)$ (that is, $r(x) = 0$)

   (ii) Prove that $f(k) = 0$

   For ($\Longleftarrow$), assume that $f(k) = 0$

   (iii) Show that $r(x) = 0$ or $\deg r(x) = 0$, where $r(x)$ is the remainder of the division of $f(x)$ by $x - k$

   Assume, towards a contradiction, that $r(x) \neq 0$. Then, we must have that $\deg r(x) = 0$.
   In that case, $r(x) = c$ for some $c \in \mathbb{R}$, where $\mathbf{c \neq 0}$

   (iv) Using the fact that $f(k) = 0$, show that $c = 0$, thus arriving to a contradiction

   (v) Conclude that $r(x) = 0$ and, therefore, $x - k$ divides $f(x)$