AP Precalculus - M3Y & M3Z

Polynomials - Homework 1

1. Let $f(x) = 2x^3 + 3x^2 - 5x + 6,\ g(x) = -2x^3 + 6x^2 - 7,\ h(x) = -x^2 - x + 4.$
   Find the following polynomials and their degrees:
   (i) $f(x) + g(x) =$ $\qquad$ $\deg\big(f(x) + g(x)\big) =$
   (ii) $f(x) - h(x) =$ $\qquad$ $\deg\big(f(x) - h(x)\big) =$
   (iii) $g(x) + h(x) =$ $\qquad$ $\deg\big(g(x) + h(x)\big) =$
   (iv) $f(x) \cdot g(x) =$ $\qquad$ $\deg\big(f(x) \cdot g(x)\big) =$
   (v) $g(x) \cdot h(x) =$ $\qquad$ $\deg\big(g(x) \cdot h(x)\big) =$

2. Find examples of polynomials $f(x)$ and $g(x)$ such that $\deg f(x) = 5,\ \deg g(x) = 5$ and
   (i) $\deg\big(f(x) + g(x)\big) = 5$
   (ii) $\deg\big(f(x) + g(x)\big) < 5$

3. Let $z, w \in \mathbb{C}$ be complex numbers. Prove the following:
   (i) $\overline{(\overline{z})} = z$
   (ii) $\overline{(z + w)} = \overline{z} + \overline{w}$
   (iii) $\overline{(z - w)} = \overline{z} - \overline{w}$
   (iv) $\overline{(z \cdot w)} = \overline{z} \cdot \overline{w}$
   (v) $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$
   (vi) If $c \in \mathbb{R}$, then $\overline{(c \cdot z)} = c \cdot \overline{z}$
   (vii) If $n \in \mathbb{N}$, then $\overline{(z^n)} = \overline{z}^n$
   (viii) If $z \in \mathbb{R}$, then $\overline{z} = z$
   (ix) If z is an imaginary number, then $\overline{z} = -z$
   (x) $z \cdot \overline{z} = \big(\mathrm{Re}(z)\big)^2 + \big(\mathrm{Im}(z)\big)^2$

4. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.
   Using the properties of the complex conjugate from Exercise 3., prove that:
   If $z \in \mathbb{C}$ and z is a root of $f(x)$ (that is, $f(z) = 0$), then $f(\overline{z}) = 0$.
   Conclude that complex roots of polynomial come in pairs of complex conjugates.

5. Find the roots of the following polynomials:
   (i) $f(x)$ if $\deg f(x) = 2$ and $f(x)$ has root $x = 5 + 3i$
   (ii) $g(x)$ if $\deg f(x) = 3$ and $g(x)$ has roots $x = -2 - i$ and $x = 1$
   (iii) $h(x)$ if $\deg h(x) = 4$ and $h(x)$ has roots $x = i$, $x = 6 + \dfrac{i}{2}$

6. Bob told Alice that his favorite polynomial $f(x)$ has $\deg f(x) = 3$ and three distinct (different) complex roots z_1, z_2, and z_3. When Alice asked him to show her that polynomial, Bob claimed that he had forgotten it at home. Alice suspects that Bob is lying. Is Alice's suspicion justified? Explain.

7. The polynomial $f(x)$ has $\deg f(x) = 5$, and roots $x = 0, x = 1 + i$, and $x = 9i$. If $f(1) = 2$, find $f(x)$.