2019

AMC 10A

DO NOT OPEN UNTIL THURSDAY, February 7, 2019

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All the information needed to administer this exam is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE FEBRUARY 7, 2019.
2. Your PRINCIPAL or VICE PRINCIPAL must verify on the AMC 10/12 COMPETITION CERTIFICATION FORM (found on **maa.org/amc** under 'AMC 10/12') that you followed all rules associated with the administration of the exam.
3. Answer sheets must be returned to the MAA AMC office the day after the competition. Ship with appropriate postage using a trackable method. FedEx or UPS is strongly recommended.
4. The publication, reproduction, or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

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MAA AMC
American Mathematics Competitions

American Mathematics Competitions

20th Annual

AMC 10A

American Mathematics Competition 10A
Thursday, February 7, 2019

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU.
2. This is a 25-question multiple-choice exam. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Sheet with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer sheet will be graded. **You must use and submit the original answer sheets provided by the MAA AMC. Photocopies will not be scored.**
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only scratch paper, graph paper, rulers, compasses, protractors, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the exam will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the exam, your competition manager will ask you to record certain information on the answer form.
8. When your competition manager gives the signal, begin working on the problems. You will have **75 minutes** to complete the exam.
9. When you finish the exam, *sign your name* in the space provided at the top of the Answer Sheet.

The Committee on the American Mathematics Competitions reserves the right to disqualify scores from a school if it determines that the required security procedures were not followed.

Students who score well on this AMC 10 will be invited to take the 37th annual American Invitational Mathematics Examination (AIME) on Wednesday, March 13, 2019, or Thursday, March 21, 2019. More details about the AIME are on the back page of this test booklet.

1. What is the value of

$$2^{\left(0^{\left(1^9\right)}\right)}+\left(\left(2^0\right)^1\right)^9?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

2. What is the hundreds digit of $(20! - 15!)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 5

3. Ana and Bonita were born on the same date in different years, n years apart. Last year Ana was 5 times as old as Bonita. This year Ana's age is the square of Bonita's age. What is n?

(A) 3 **(B)** 5 **(C)** 9 **(D)** 12 **(E)** 15

4. A box contains 28 red balls, 20 green balls, 19 yellow balls, 13 blue balls, 11 white balls, and 9 black balls. What is the minimum number of balls that must be drawn from the box without replacement to guarantee that at least 15 balls of a single color will be drawn?

(A) 75 **(B)** 76 **(C)** 79 **(D)** 84 **(E)** 91

5. What is the greatest number of consecutive integers whose sum is 45?

(A) 9 **(B)** 25 **(C)** 45 **(D)** 90 **(E)** 120

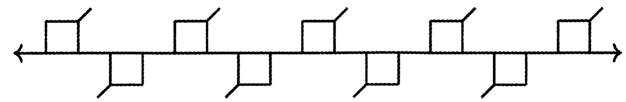
6. For how many of the following types of quadrilaterals does there exist a point in the plane of the quadrilateral that is equidistant from all four vertices of the quadrilateral?
   - a square
   - a rectangle that is not a square
   - a rhombus that is not a square
   - a parallelogram that is not a rectangle or a rhombus
   - an isosceles trapezoid that is not a parallelogram

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

7. Two lines with slopes $\frac{1}{2}$ and 2 intersect at $(2,2)$. What is the area of the triangle enclosed by these two lines and the line $x+y=10$?

(A) 4 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** 8 **(E)** $6\sqrt{2}$

8. The figure below shows line ℓ with a regular, infinite, recurring pattern of squares and line segments.



How many of the following four kinds of rigid motion transformations of the plane in which this figure is drawn, other than the identity transformation, will transform this figure into itself?
   - some rotation around a point on line ℓ
   - some translation in the direction parallel to line ℓ
   - the reflection across line ℓ
   - some reflection across a line perpendicular to line ℓ

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

9. What is the greatest three-digit positive integer n for which the sum of the first n positive integers is <u>not</u> a divisor of the product of the first n positive integers?

(A) 995 **(B)** 996 **(C)** 997 **(D)** 998 **(E)** 999

10. A rectangular floor that is 10 feet wide and 17 feet long is tiled with 170 one-foot square tiles. A bug walks from one corner to the opposite corner in a straight line. Including the first and last tile, how many tiles does the bug visit?

(A) 17 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

11. How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

(A) 32 **(B)** 36 **(C)** 37 **(D)** 39 **(E)** 41

12. Melanie computes the mean μ, the median M, and modes of the 365 values that are the dates in the months of 2019. Thus her data consist of 12 1s, 12 2s, ..., 12 28s, 11 29s, 11 30s, and 7 31s. Let d be the median of the modes. Which of the following statements is true?

(A) $\mu < d < M$ **(B)** $M < d < \mu$ **(C)** $d = M = \mu$
(D) $d < M < \mu$ **(E)** $d < \mu < M$

13. Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40^\circ$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

(A) 90 **(B)** 100 **(C)** 105 **(D)** 110 **(E)** 120

14. For a set of four distinct lines in a plane, there are exactly N distinct points that lie on two or more of the lines. What is the sum of all possible values of N?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 19 **(E)** 21

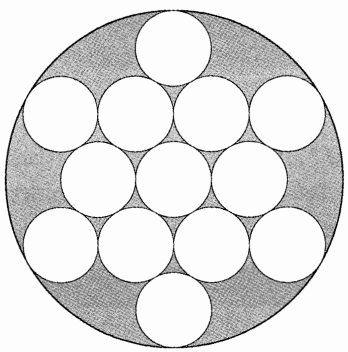
15. A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

16. The figure below shows 13 circles of radius 1 within a larger circle. All the intersections occur at points of tangency. What is the area of the region, shaded in the figure, inside the larger circle but outside all of the circles of radius 1?



(A) $4\pi\sqrt{3}$ **(B)** 7π **(C)** $\pi(3\sqrt{3}+2)$ **(D)** $10\pi(\sqrt{3}-1)$
(E) $\pi(\sqrt{3}+6)$

17. A child builds towers using identically shaped cubes of different colors. How many different towers with a height of 8 cubes can the child build with 2 red cubes, 3 blue cubes, and 4 green cubes? (One cube will be left out.)

(A) 24 **(B)** 288 **(C)** 312 **(D)** 1,260 **(E)** 40,320

18. For some positive integer k, the repeating base-k representation of the (base-ten) fraction $\frac{7}{51}$ is $0.\overline{23}_k = 0.23232323\ldots_k$. What is k?

(A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 17

19. What is the least possible value of

$$(x+1)(x+2)(x+3)(x+4) + 2019,$$

where x is a real number?

(A) 2017 **(B)** 2018 **(C)** 2019 **(D)** 2020 **(E)** 2021

20. The numbers $1, 2, \ldots, 9$ are randomly placed into the 9 squares of a 3×3 grid. Each square gets one number, and each of the numbers is used once. What is the probability that the sum of the numbers in each row and each column is odd?

(A) $\frac{1}{21}$ **(B)** $\frac{1}{14}$ **(C)** $\frac{5}{63}$ **(D)** $\frac{2}{21}$ **(E)** $\frac{1}{7}$

21. A sphere with center O has radius 6. A triangle with sides of length 15, 15, and 24 is situated in space so that each of its sides is tangent to the sphere. What is the distance between O and the plane determined by the triangle?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $3\sqrt{2}$ **(D)** $2\sqrt{5}$ **(E)** 5

22. Real numbers between 0 and 1, inclusive, are chosen in the following manner. A fair coin is flipped. If it lands heads, then it is flipped again and the chosen number is 0 if the second flip is heads, and 1 if the second flip is tails. On the other hand, if the first coin flip is tails, then the number is chosen uniformly at random from the closed interval $[0, 1]$. Two random numbers x and y are chosen independently in this manner. What is the probability that $|x - y| > \frac{1}{2}$?

(A) $\frac{1}{3}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{2}{3}$

23. Travis has to babysit the terrible Thompson triplets. Knowing that they love big numbers, Travis devises a counting game for them. First Tadd will say the number 1, then Todd must say the next two numbers (2 and 3), then Tucker must say the next three numbers (4, 5, 6), then Tadd must say the next four numbers (7, 8, 9, 10), and the process continues to rotate through the three children in order, each saying one more number than the previous child did, until the number 10,000 is reached. What is the 2019th number said by Tadd?

(A) 5743 **(B)** 5885 **(C)** 5979 **(D)** 6001 **(E)** 6011

24. Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. There exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all real numbers s with $s \notin \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

(A) 243 **(B)** 244 **(C)** 245 **(D)** 246 **(E)** 247

25. For how many integers n between 1 and 50, inclusive, is

$$\frac{(n^2-1)!}{(n!)^n}$$

an integer? (Recall that $0! = 1$.)

(A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

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Questions and comments about problems and solutions for this exam should be sent to:

amchq@maa.org

Send questions and comments about administrative arrangements to:

amcinfo@maa.org

or

MAA American Mathematics Competitions
P.O. Box 471
Annapolis Junction, MD 20701

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

2019 AIME

The 37th annual AIME will be held on Wednesday, March 13, 2019, with the alternate on Thursday, March 21, 2019. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate if you achieve a high score on this competition. Top-scoring students on the AMC 10/12/AIME will be selected to take the 48th Annual USA Mathematical Olympiad (USAMO) on April 17–18, 2019.